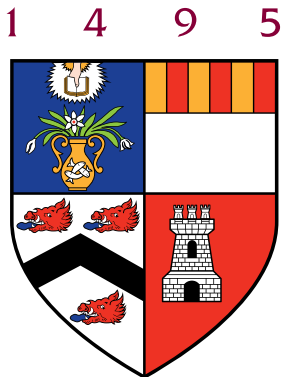

Collective behavior in complex oscillatory systems

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Declaration

I, Pau Clusella Coberó, hereby declare that this thesis has been composed by me and it is my own work, with joint collaborations specified in the corresponding chapters. I also declare that this thesis has not been presented in any previous application for a degree, diploma, or other similar award; and that all sources of information have been specifically acknowledged and all external quotations distinguished by quotation marks.

Abstract

In this thesis we analyze different instances of dynamical regimes emerging from large ensembles of coupled oscillatory units. On one side, we study systems of homogeneous globally coupled oscillators, where a macroscopic treatment at the thermodynamic limit is possible. In these setups our main interest is to characterize the emergence and stability of the so-called *self-consistent partial synchrony* (SCPS), a family of collective states where the evolution of the single units differs from that of the macroscopic observables. First we show that a simple form of SCPS emerges in a Kuramoto-Daido phase model where the interaction among the oscillators is set by a coupling function with two harmonics. Invoking the associated nonlinear Liouville operator we are able to characterize semi-analytically the shape and stability properties of such regime. Then, we show that a small stochastic force acting independently on each oscillator of the biharmonic model is able to stabilize SCPS to the entire parameter range. Such noise-induced bifurcation is treated semi-analytically by extending the macroscopic analysis of the deterministic model to the solution of the corresponding nonlinear Fokker-Plank equation through a perturbative approach. Next, we investigate more complex forms of SCPS, including collective chaos, emerging in a system of mean-field coupled amplitude oscillators where the single units spontaneously distribute along a smooth time-varying closed curve, thereby maintaining the typical ordering of purely phase oscillators. We develop a formalism to treat this situation macroscopically by means of two nonlinear partial integro-differential equations. Finally, we move to systems coupled through irregular topologies. In particular, we construct two different methods to modify the coupling structure while preserving the original dynamical behavior. This is achieved through the identification of the degrees of freedom

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responsible for the emergence of the dynamical pattern on the Laplacian of the network.

Summary

This thesis studies different instances of systems composed of many coupled oscillatory units showing emerging global behavior. Here we summarize the main findings and salient points.

First, we study ensembles of homogeneous globally coupled phase oscillators, our main interest being the emergence and stability properties of self-consistent partial synchrony (SCPS), a family of dynamical regimes where the evolution of the single units differs from that of the global field:

- We show that a second harmonic on the coupling function of the otherwise homogeneous Kuramoto-Sakaguchi model suffices to display periodic SCPS.
- The shape and stability of the probability density of the phases is studied at the thermodynamic limit through the corresponding nonlinear Liouville operator.
- Both, macroscopic analysis and microscopic simulations agree on determining the emergence of SCPS from the splay state and its loss of stability through a subcritical Hopf bifurcation.
- Self-consistent partial synchrony arises in a similar fashion in an ensemble of weakly-coupled Rayleigh oscillators.
- Phase reduction of the Rayleigh model provides a coupling function of the Kuramoto-Daido model with eight harmonics that retains the behavior of the original system.

Summary

Second we study the effect of a small stochastic force acting independently on each oscillator of the biharmonic model:

- A small level of noise suffices to stabilize SCPS throughout the entire parameter range of interest.
- The analytical treatment of SCPS in the stochastic model is attained by extending the results of the deterministic case to the nonlinear Fokker-Plank equation of the stochastic model by means of a perturbative approach.
- Macroscopic and microscopic results coincide to describe the stabilization of SCPS through a supercritical Hopf bifurcation.
- A second bifurcation line corresponding to a saddle of periodic orbits bounding the bistable region is found with the help of numerical simulations.
- The noise-induced stabilization of SCPS materializes also on the ensemble of Rayleigh oscillators studied in the deterministic case.

Next, we move to a globally coupled ensemble of Stuart-Landau oscillators, where the single units are described by two variables. We focus on a parameter range where the oscillators spontaneously distribute themselves on a smooth closed curve that evolves in time.

- A mathematical formalism to treat this situation is developed at the thermodynamic limit, resulting on a system of two nonlinear partial integro-differential equations.
- Microscopic and macroscopic simulations of the system coincide to unveil a set of bifurcations from splay state to different forms of SCPS and, finally, collective chaos.
- Splay state and periodic SCPS can be treated analytically with the help of the new macroscopic equations.

-
- We study mathematically the emergence of SCPS out of the splay state and confirm that the presence of higher harmonics in the coupling is necessary for SCPS to emerge.
 - Poincaré sections unveil two other forms of SCPS where the collective behavior is ruled by two and three leading frequencies respectively.
 - A new transition to collective chaos is found with the help of Lyapunov analysis at both, microscopic and collective level.
 - Multifractal analysis of the transverse Lyapunov exponent confirms that the single units are non-chaotic even though the macroscopic observables display high-dimensional chaos.

Finally, we study homogeneous reaction-diffusion systems defined on complex network topologies. In particular we tackle the problem of creating different network topologies that display the same dynamical behavior. We focus on regimes arising from perturbing the unstable homogeneous equilibrium.

- Preserving the unstable directions usually is enough to reproduce the original dynamics of the system.
- In those cases where the eigenvectors of the Laplacian are highly localized on a subset of nodes, the topology structure of the modified network might present important changes.
- Localization properties allow to implement changes already at the level of the adjacency matrix, providing thus a major control of the modifications.
- In general, it is possible to modify the network topology while preserving the collective behavior of the system.

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This PhD thesis is the result of three years of scientific research at the University of Aberdeen and University of Florence. Therefore, it contains original results obtained by the author and co-workers from October 2015 to September 2018. Most of the content included in this dissertation is either published in scientific journals or in review process. Although I modified substantial parts of each chapter to present a consistent and coherent work, many paragraphs have been directly extracted from the main reference paper or preprint. Here I list the relation between the chapters of the thesis and the corresponding publication of reference:

- Chapter 2: Pau Clusella, Antonio Politi, and Michael Rosenblum. A minimal model of self-consistent partial synchrony. *New Journal of Physics*, 18(9):093037, 2016.
- Chapter 3: Pau Clusella and Antonio Politi. Noise-induced stabilization of collective dynamics. *Phys. Rev. E*, 95:062221, 2017.
- Chapter 4: Pau Clusella and Antonio Politi. From phase to amplitude oscillators. *ArXiv e-prints*, 2018.
- Chapter 5: Giulia Cencetti, Pau Clusella, and Duccio Fanelli. Pattern invariance for reaction-diffusion systems on complex networks. *Scientific Reports*, 8:16226, 2018.

Also, during this period I had the chance to work on a side project about network immunization, which have been left out of this dissertation for lack of consistency with the main topic of the thesis. The reference follows:

Bibliographical note

- Pau Clusella, Peter Grassberger, Francisco J. Pérez-Reche, and Antonio Politi. Immunization and targeted destruction of networks using explosive percolation. *Phys. Rev. Lett.*, 117:208301, 2016

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Introduction

Dynamical systems provide a formal tool to model the behavior of elements or quantities that evolve in time. Indeed, a simple equation suffices to faithfully characterize, for instance, the orbit of a planet around the sun [1] or the voltage oscillations of an isolated spiking neuron [2]. Nevertheless, when considering these units as a part of a whole –the solar system or a layer of brain cortex– the scenario changes. Without the need of an established hierarchy or purpose, the larger system might spontaneously display emergent properties that cannot be traced back to a simple superposition of the single units. This type of systems are generically referred to as *complex systems* [3, 4].

One of the main characteristics of complex systems consists in the emergence of spontaneous order through multiple scales. Even in the simple case of many identical interacting elements one can distinguish at least two different types of dynamics: the *collective* (macroscopic) behavior versus that of the *single* (microscopic) units. The analysis of macroscopic states arising in systems composed of many units is a well established topic in mathematics and physics. Statistical mechanics offers a complete picture for systems at equilibrium, but a theory for non-equilibrium systems is missing. Understanding the emergence of collective dynamical regimes in large systems is, in fact, a mathematical challenge since one needs to deal with many degrees of freedom. Moreover, this area of research has grown in the last decades since the increased performance of computers allows to attain many aspects of such systems using numerical simulations.

Synchronization phenomena in networks of coupled oscillatory units represents one of

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the simplest instances of collective behavior, and yet, proves significant in many areas of research such as biology, neuroscience or climate systems [5]. The most trivial form of synchronization corresponds to states where all the microscopic units evolve identically, so that the collective motion is nothing else than a macroscopic representation of the single cells. Conceptually opposed are the incoherent states, where the oscillators evolve independently from one another, so that the collective variables either relax to a steady state possibly affected by random fluctuations. Nevertheless, it is unclear what and under which conditions other types of complex behavior are to be expected. In general, one needs to account for (i) the dynamical system that rules the evolution of each single unit (microscopic dynamics), (ii) the type of interaction between the different units of the system (shape of coupling), and (iii) the structure of the interactions (network topology). In this thesis we explored different aspects concerning these three characteristics in order to study -numerically and analytically- distinct forms of collective behavior in networks of coupled oscillators.

The amplitude of a periodic oscillator disturbed by small perturbations can be adiabatically eliminated, so that each unit is characterized by a single variable: the *phase* (see chapter 1 for a detailed explanation). Phase models represent, therefore, the simplest instance of complex oscillatory systems. The famous Kuramoto-Sakaguchi model [6]—paradigm of collective synchronization—consists of a phase model where the interaction between the units is purely sinusoidal. On its original formulation, each oscillator interacts with all the other units in the same manner: the system is *globally coupled*. In practice, this means that an individual oscillator is both, subject and contributing to a *mean-field* variable that represents the collective dynamics. This simple coupling structure allows for a mathematical analysis in the thermodynamic limit by considering the probability density of the phases as a global functional (*i.e.*, infinite-dimensional) variable describing an ensemble with infinite oscillators. In this collective framework, it is possible to distinguish between two different types of solutions: those where the oscillators are *locked* within the distribution, and those where microscopic units display a different behavior from that of the global field, referred to as *self-consistent partial synchrony* (SCPS) (see chapter 2 for

a more complete definition). If there is no diversity among the single units, the emergence of dynamical regimes without phase locking phenomena is far from trivial. Indeed, the homogeneous Kuramoto-Sakaguchi model only displays either full synchrony or incoherent states. Moreover, the Watanabe-Strogatz theory provides a low-dimensional description of the Kuramoto-Sakaguchi model in terms of three collective variables and infinitely-many constants of motion [7, 8, 9, 10]. Thus, the sinusoidal coupling is not representative of the dynamical richness that globally coupled complex oscillatory systems might present.

The simplest form of SCPS corresponds to the case where the macroscopic dynamics displays purely periodic motion, but the single oscillators evolve quasiperiodically. This situation was first observed in an ensemble of leaky integrate-and-fire (LIF) neurons [11] and later on, in the associated phase model [12]. Other cases of SCPS have been obtained with nonlinear coupling [13] or delayed interactions [14], but the minimal conditions for its appearance in phase models have been unknown. The first original work contained in this thesis, presented in chapter 2, addresses this question by studying the emergence and stability of self-consistent partial synchrony in a simple model of phase oscillators. We consider a coupling function characterized by two harmonics, as studied in [15], breaking thus the symmetry of the otherwise sinusoidal coupling. By invoking the thermodynamic limit we are able to characterize the shape of the probability distribution of the phases as a stationary solution of the corresponding (nonlinear) Liouville operator in a rotating frame. Linear stability analysis also shows, in agreement with the numerical simulations, that this state loses stability through a subcritical Hopf bifurcation. Although we focus on SCPS, we also review in detail other dynamical regimes displayed by the model: synchronization, incoherent states, clustered states, and an interesting switching behavior through an heteroclinic cycle between clusters. The final part of this chapter is devoted to prove the ubiquity of self-consistent partial synchrony by studying its appearance in an ensemble of Rayleigh oscillators and its corresponding phase reduction.

There are two simple ways to challenge the homogeneity of the single oscillators. One of them, used in the original Kuramoto model, is to consider different natural frequencies

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for each uncoupled oscillator, adding thus *quenched heterogeneity* in the ensemble. The second consists in adding independent stochastic forces to the evolution of each oscillator. Quenched heterogeneity difficulties the comparison across scales: a single oscillator might not be entrained to the global distribution either because of the intrinsic nature of the collective oscillation or because its natural frequency differs too much from the global one. Instead, random fluctuations still allow to consider all the oscillators as *identical*, so that phase locking phenomena affects all units in the same manner. On the other hand, stochastic forces are unavoidably present in physical systems. Thus, in chapter 3 we illustrate the effect of a small source of thermal noise acting independently on each oscillator of the biharmonic model. The small random fluctuations have the counter-intuitive effect of stabilizing SCPS to the entire parameter range, illustrating thus a novel example of dynamical order emerging out from disordered units. Numerical simulations at the microscopic level are again complemented with a macroscopic description, this time given by a nonlinear Fokker-Plank equation, to unveil the noise-induced bifurcation leading to stable SCPS.

Going one step further in terms of collective complexity, in chapter 4 we address the question whether systems of identical phase oscillators can display collective chaos. Most of the chaotic states studied so far in mean-field coupled oscillatory systems consist of regimes where the amplitude of the oscillators plays an important role, so that it cannot be adiabatically eliminated [16, 17, 18]. Nevertheless, Nakagawa and Kuramoto reported on a chaotic state where the oscillators distribute themselves along a closed, smooth, time-dependent curve, thereby maintaining the typical ordering of (identical) phase oscillators [19]. We study this phenomena, which has been so far overlooked, by developing a general formalism based on two partial differential equations. One of them describes the evolution of the probability density along the closed curve, as in systems of phase oscillators. The second equation models the shape of the curve itself. The formalism is specifically developed for Stuart-Landau oscillators, but it is general enough to be applied to other classes of amplitude oscillators. Using these equations we are able to identify and study a set of bifurcations going from periodic SPCS to collective chaos, passing through various forms of SCPS with different complexity. In fact, the chaotic state can be seen as

a generalization of a SCPS state characterized by a multifractal probability density. Combining a Lyapunov analysis at both, macro and microscopic level we shed further light on the relationship between the two levels of description.

Finally, in the last chapter of the thesis we leave behind globally coupled systems to study the effect of irregular network topologies on the collective states. Complex network analysis is a major topic in the study of large dynamical systems [20]. In general, the relationship between a given architecture of connections and the overall function of the system is unknown. Nevertheless, homogeneous reaction-diffusion systems always admit a full synchronous state whose stability can be addressed by the diagonalization of the network Laplacian [21]. Thus, in this case there is indeed an explicit dependence between the dynamics of the equilibrium state and the network topology. If the synchronous solutions proves unstable, any small perturbation develops irregular spatio-temporal patterns. We exploit this fact to construct two different methods that modify the connections of a given network while maintaining the dynamical behavior of the system. Although we focus again on Stuart-Landau units, the method is sufficiently general to be applied to other models beyond oscillatory behavior. We provide an example of this situation with the Brusselator model.

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Chapter 1

Introduction to phase oscillators

Oscillations are ubiquitous in nature. Some examples are the dynamics of astronomic objects [1], AC circuits [2], the behavior of spiking neurons [3], heart beats [4], circadian rhythms [5], climate models [6], economic cycles [7], and chemical reactions [8]. The list could go on and on. Hence, the behavior and properties of oscillatory systems constitutes a vast topic in the scientific literature. In the theory of dynamical systems, a second order ODE is enough to display purely periodic motion. In fact, after fixed points, limit-cycles are the simplest invariant solutions one can find in a given system.

Mainly motivated by the study of biological systems, Arthur Winfree was the first scientist to investigate collective behavior in oscillatory systems [9, 10]. He showed that, despite the great variety of systems displaying periodic motion, under proper assumptions it is possible to reduce the behavior of an oscillator subject to external perturbations to the dynamics of a single variable, the phase. Later on, Kuramoto used averaging techniques to reduce the Winfree model to a simpler class of systems where the interaction between the oscillators is set solely by a coupling function [11].

Therefore, phase models provide an important framework to study the self-adjustment of rhythms in systems composed by many interacting periodic oscillators. The simplest form of such collective behavior, and yet, ubiquitous in nature, is synchronization phe-

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nomena [12]. Synchronization can occur already in systems with only two interacting oscillators. In fact, one of the firsts scientific notes about synchronization, although highly anecdotal, consists on the phase locking of two mechanical clocks by Huygens. In larger systems, the Kuramoto model represents a paradigmatic model for the study of synchronization transitions. In this case, the shape of the interaction between oscillators is purely sinusoidal so that the mean-field coupling is through a single global parameter: the Kuramoto order parameter. Nevertheless, the Kuramoto model is nowadays well understood and, moreover, the Watanabe-Strogatz theory provides a full macroscopic description of its behavior [13, 14, 15, 16]. In this thesis we go beyond such theory, first by using more harmonics in the coupling function (Chapters 2 and 3), then relaxing the concept of "phase oscillator" to a case where the amplitude cannot be adiabatically eliminated (Chapter 4), and finally considering generic systems coupled through irregular architectures (Chapter 5).

In the remaining of this introduction we review the phase reduction method that leads to a phase model of Kuramoto-Daido type. We then provide an exemplary analysis of a single periodically forced oscillator to illustrate important concepts such as phase-locking and Arnold tongues. Next we illustrate the Kuramoto-Sakaguchi system as simple instance of collective synchronization. Finally, we briefly show that, at the thermodynamic limit, the Kuramoto-Sakaguchi system can be mapped to a low-dimensional system by means of the Ott-Antonsen Ansatz, which falls into the Watanabe-Strogatz theory.

The results showed in this chapter are well established in the literature. For basic concepts on phase oscillators and synchronization we have mainly followed [17, 12, 11]. The first of these references consists of a short and pedagogical introductory chapter to phase reduction and synchronization. The second is a more complete, but also introductory, work on phase oscillators and synchronization phenomena. Kuramoto's book [11] provides instead a more technical analysis on the phase reduction method. For the Watanabe-Strogatz theory and the Ott-Antonsen Ansatz we mainly followed the original papers and later revisions [13, 14, 15, 16].

1.1 Oscillatory systems

In this section we briefly introduce three different examples of systems displaying oscillation phenomena.

1.1.1 FitzHugh-Nagumo model

An important class of oscillatory systems used nowadays in scientific research are those used in computational neuroscience to model the dynamics of a spiking neuron. In 1952 Hodgkin and Huxley presented a system of four nonlinear Ordinary Differential Equations (ODEs) describing the action potential of the axon of a giant squid [3]. This work granted them the Nobel Prize in Physiology or Medicine 1963. In 1963 and 1964 two independent works by FitzHugh and Nagumo provided a simplification of the Hodgkin-Huxley model to two ODEs that properly reproduce the behavior of a spiking neuron under external input. The FitzHugh-Nagumo model reads [18]

$$\begin{aligned}\dot{x} &= x - \frac{x^3}{3} - y + I, \\ \tau \dot{y} &= (x + a - by),\end{aligned}$$

where x is the voltage potential of the neuron, and y is a variable that accounts for the recovery of the different ion channels. The parameter I is the external input current, and $\tau = 0.08$ indicates that the time scale of y is slower than x . The other parameters of the recovery equations are set to $a = 0.7$, and $b = 0.8$. If the external current I arriving to the neuron is smaller than a certain threshold, I_c , the neuron is at rest in a fixed point. At $I = I_c$ there is a transition above which the system oscillates in a periodic manner around a stable limit-cycle: the neuron is spiking. Figure 1.1(a1) shows the stable periodic orbit of the FitzHugh-Nagumo model for $I = 0.5$ (black curve), and a trajectory converging to it (red curve). The periodicity of the attractor becomes more evident in Figure 1.1(a2), which displays the time series of the x and y variables along the limit-cycle for 4 periods.

1.1.2 Rayleigh oscillator

Another well-known system displaying periodic behavior is the Rayleigh oscillator, which models the vibration of a reel producing sound waves [19]. In this case the system reads as the second order ODE,

$$\ddot{x} - \zeta(1 - \dot{x}^2)\dot{x} + \omega^2 x = 0,$$

which using the transformation $\dot{x} = y$ can be mapped to the two-dimensional system

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= \zeta(1 - y^2)y - \omega^2 x.\end{aligned}$$

Similarly to the FitzHugh-Nagumo model, the Rayleigh oscillator has a fixed point that loses stability upon increasing ζ . For $\zeta = 5$ the system displays the limit cycle depicted in Figures 1.1(b1) and (b2).

1.1.3 Stuart-Landau system

As many other oscillatory systems, both the FitzHugh-Nagumo limit-cycle and the Rayleigh oscillator emerge after a stationary fixed point loses stability through a supercritical Hopf-bifurcation upon changing the parameters of the model. Thus, close to the transition point, the behavior of the system can be described by a *normal form*, a reduced dynamical system that captures the essential properties of the original model close to the bifurcation. This is possible through retaining the fast terms of the original system and eliminating the slower variables adiabatically via the *center manifold theorem* [11].

The normal form of a Hopf bifurcation receives the name of Stuart-Landau equation, which, for the supercritical case, reads

$$\dot{z} = z - (1 + i\alpha)|z|^2 z$$

1.1. Oscillatory systems

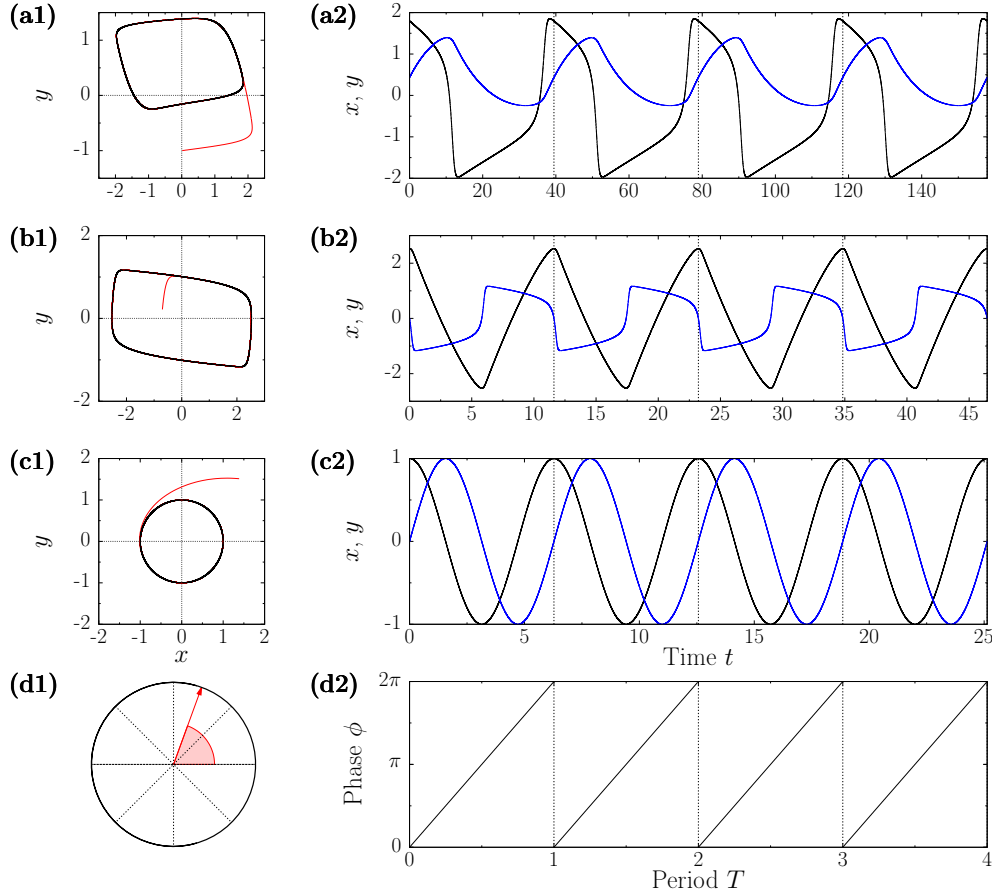


Figure 1.1: Periodic behavior of the FitzHugh-Nagumo system (a1,a2), the Rayleigh oscillator (b1,b2), and the Stuart-Landau equation (c1,c2). (a1), (b1), and (c1) show the attracting limit-cycle (black curve) on the (x, y) -plane, and an exemplary trajectory converging to it (red curve). (a2), (b2), and (c2) show the time series of x and y on the limit cycle for 4 periods. (d1) depicts a phase variable on the torus, (d2) shows the linear growth of the phase of an oscillator with period T .

where $z = x + iy \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ is a parameter that accounts for the frequency of rotation on the limit cycle. In fact, the limit-cycle of the Stuart-Landau equation has the simple form $z(t) = e^{i\alpha t}$. Figure 1.1(c1) displays such limit-cycle on the complex plane, and Figure 1.1(c2) the corresponding time-series.

1.2 Phase reduction

The three systems presented before provide a good insight about the variety of shapes and periods that limit-cycles might exhibit. Nevertheless, under proper assumptions it is possible to reduce the behavior of these systems using a single variable: the phase of the oscillator. This reduction allows to study oscillatory systems in a generic way and, moreover, provides the foundations to study emergent phenomena on systems composed of many interacting oscillatory units.

Let us consider an m -dimensional variable $\mathbf{x}$ (typically, one can think of $m = 2$, so that $\mathbf{x} = (x, y)$), whose time evolution is ruled by the differential equation

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) \quad (1.1)$$

where $\mathcal{F}$ is a smooth function. Let us assume that equation (1.1) has an attracting limit-cycle with period T , $\mathbf{x}_{\text{LC}}(t) = \mathbf{x}_{\text{LC}}(t + T)$. Thus, the curve $\mathbf{x}$ is, by definition, parameterized by time, $t \in [0, T)$. We define the *phase* of a point $\mathbf{x}(t)$ of $\mathbf{x}_{\text{LC}}$ as

$$\phi(\mathbf{x}(t)) := t \frac{2\pi}{T} \in [0, 2\pi).$$

For simplicity, we denote $\omega := \frac{2\pi}{T}$.

The concept of phase is well defined for any point lying exactly on the limit cycle. Nevertheless, the phase reduction approach is based on extending the notion of phase to a neighbourhood G of $\mathbf{x}_{\text{LC}}$. Let $\mathbf{y}_0 = \mathbf{y}(0)$ be a point in the state space close to, but not exactly on the periodic attractor $\mathbf{x}_{\text{LC}}$. Since the limit cycle is stable, the trajectory $\mathbf{y}(t)$ will end up arbitrarily close to $\mathbf{x}_{\text{LC}}$ as $t \rightarrow \infty$, so that it will have a well defined phase, $\phi(\mathbf{y}(t))$. Therefore, even if $\mathbf{y}_0$ is not on the invariant manifold, one can find a point, $\mathbf{x}_0 \in \mathbf{x}_{\text{LC}}$ so that at long-term, $\phi(\mathbf{y}(t)) = \phi(\mathbf{x}(t))$. One can then define the phase of $\mathbf{y}_0$ as $\phi(\mathbf{y}_0) = \phi(\mathbf{x}_0)$. All the points in the neighbourhood of $\mathbf{x}_{\text{LC}}$ associated to the same phase form a *isochrone curve*.

Formally, the concept of isochrone can be derived as follows. In general, one can express the time evolution of a phase as

$$\dot{\phi}(\mathbf{x}) = \omega, \quad \mathbf{x} \in G. \quad (1.2)$$

On the other hand, applying the chain rule and using Eq. 1.1, one obtains

$$\dot{\phi}(\mathbf{x}) = \text{grad}_{\mathbf{x}} \phi \cdot \mathcal{F}(\mathbf{x}).$$

Combining the two previous equations provides

$$\text{grad}_{\mathbf{x}} \phi \cdot \mathcal{F}(\mathbf{x}) = \omega. \quad (1.3)$$

The gradient of ϕ expresses the variation of a phase with respect to space, thus, the *isochrones* are nothing else than the level curves of $\text{grad}_{\mathbf{x}} \phi$. This equation expresses then, that the variation of the phase under the effect of the field $\mathcal{F}$ is constant. Figures 1.2(a1), (b1), and (c1) show isochrones of the FitzHugh-Nagumo, Rayleigh, and Stuart-Landau systems respectively (see red curves).

We want to study oscillatory systems that are interacting with a certain environment, so, usually, equation (1.1) is not enough for our purposes. One needs to account for the effect that external perturbations have on the system. Let us assume a perturbation on the evolution of $\mathbf{x}$, $\mathbf{p}(t)$, with a small amplitude parameter $0 < \varepsilon \ll 1$,

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) + \varepsilon \mathbf{p}(t).$$

In order to keep arguing in terms of phases, one needs to find which is the impact of such perturbation on the phase of the oscillator. Applying again the chain rule to the time evolution of ϕ , and using Eq. (1.3) one obtains

$$\dot{\phi}(\mathbf{x}) = \text{grad}_{\mathbf{x}} \phi \cdot (\mathcal{F}(\mathbf{x}) + \varepsilon \mathbf{p}(t))$$

Chapter 1. Introduction to phase oscillators

$$= \omega + \varepsilon \operatorname{grad}_{\mathbf{x}} \phi \cdot \mathbf{p}(t) \quad (1.4)$$

Thus, as one would expect, the linear response of the phase ϕ to a perturbation of the trajectory is given by its gradient. For this reason, the gradient of the phase along a certain unitary direction $\delta \mathbf{x}$ is usually referred to as the *Phase Response Curve* (PRC), $Z(\phi) := \operatorname{grad}_{\mathbf{x}} \phi \cdot \delta \mathbf{x}$. Following the definition of gradient, given a infinitesimal perturbation $\varepsilon \delta \mathbf{x}$ of a point on the limit cycle $\mathbf{x}$, one can effectively compute the phase response curve as the normalized difference between the phase of $\mathbf{x}$, and that of $\mathbf{x} + \varepsilon \delta \mathbf{x}$,

$$Z(\phi) = \frac{\phi(\mathbf{x}) - \phi(\mathbf{x} + \varepsilon \delta \mathbf{x})}{\varepsilon} .$$

Thus, a positive (resp. negative) PRC indicates that the perturbation has the effect of advancing (resp. delaying) the phase of the oscillator. A zero PRC means that the perturbed point lies exactly on the isochrone of the original point. Figures 1.2(a2),(b2), and (c2) show the PRC of the of the FitzHugh-Nagumo, Rayleigh, and Stuart-Landau oscillators for a infinitesimal perturbation along the x -direction, $\delta \mathbf{x} = (1, 0)$ (black curves), and $\delta \mathbf{x} = \frac{1}{\sqrt{2}}(1, 1)$ (blue curves). For simplicity, on the following we assume that the perturbation $\mathbf{p}(t)$ has direction $\delta \mathbf{x}$ and modulus $p(t)$ so that $\operatorname{grad}_{\mathbf{x}} \phi \cdot \mathbf{p}(t) = Z(\phi)p(t)$. Thus, Eq (1.4) now reads

$$\dot{\phi}(\mathbf{x}) = \omega + \varepsilon Z(\phi)p(t) . \quad (1.5)$$

Systems with this form are known as *Winfree-type models* of phase oscillators [9], where the PRC, $Z(\phi)$, and the shape of the forcing $\mathbf{p}(t)$ are defined independently one from the other. Nevertheless, it is possible to further simplify this system by separating time scales on the dynamics of the phase model. In order to clarify the explanation, let's assume now that $p(t)$ is a periodic function with period $T' = 2\pi/\Omega$.

The unperturbed phase oscillator described by Eq. (1.2) has the solution

$$\phi = \omega t + \phi_0 .$$

1.2. Phase reduction

If $\varepsilon \ll 1$ in Eq. (1.5), one expects that the changes in the evolution of ϕ mostly happen at a slow time scale, since the fast dynamics is driven by the natural frequency of oscillation ω . Let's study then the time evolution of the variable $\varphi = \phi - \omega t$. With this change of variables, equation (1.4) now reads

$$\dot{\varphi} = \varepsilon Z(\varphi - \omega t)p(t)$$

so that the dynamics of the system is purely governed by the coupling term. In order to retain the terms relevant for the evolution of φ one can expand the previous equation in Fourier series,

$$\begin{aligned} \dot{\varphi} = \varepsilon Z(\varphi - \omega t)p(t) &= \frac{\varepsilon}{4\pi^2} \sum_{m,n} \tilde{Z}(m)\tilde{p}(n)e^{-i[m(\varphi-\omega t)+n\Omega]t} \\ &= \frac{\varepsilon}{4\pi^2} \sum_{m,n} \tilde{Z}(m)\tilde{p}(n)e^{-im\varphi}e^{i(m\omega-n\Omega)t}. \end{aligned} \quad (1.6)$$

The last expression provides a separation of the fast and slow terms in the series: the terms with

$$m\omega - n\Omega \simeq 0$$

are resonant, and, therefore, lead to slow dynamics. The other terms, instead, have explicit dependence on t , and, therefore, lead to rapid changes. Thus, in order to keep only the relevant dynamics of the system, one can average out the non-resonant terms. Under the assumption that the natural frequency of the oscillator, ω , compares to the mean frequency of the forcing, Ω , i.e., $\omega \simeq \Omega$, the terms of Eq. (1.6) relevant for the dynamics are those with $n = m$. Thereby formally one can define a *coupling function*

$$G(\varphi) := \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \tilde{Z}(-n)\tilde{p}(n)e^{-in\varphi} = \int_0^{2\pi} d\psi Z(\psi - \varphi)p(\psi)$$

so that system (1.5) now reads

$$\dot{\phi} = \omega + \varepsilon G(\Omega t - \phi).$$

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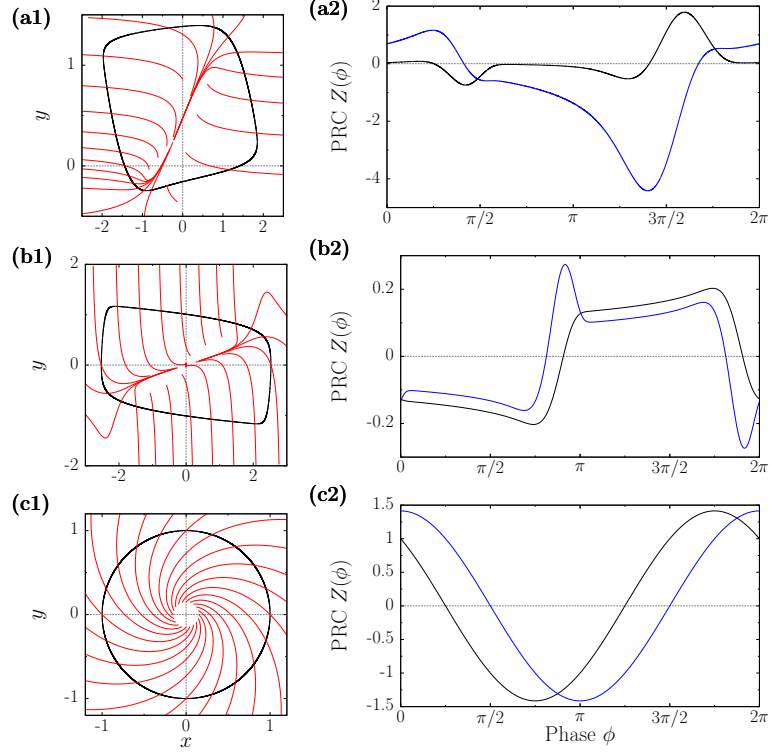


Figure 1.2: (a1), (b1), and (c1): isochrones of the FithHugh-Nagumo model, the Rayleigh oscillator, and the Stuart-Landau equation (red curves). The isochrones have been computed starting from 20 different points distributed uniformly along the phase of the oscillator. (a2), (b2), and (c2): Phase response curve of the same systems, computed for a perturbation of the direction of $(x, y) = (1, 0)$ (black curve), and $(x, y) = \frac{1}{2}(1, 1)$ (blue curve).

Phase oscillators ruled by equations of this type are known as *Kuramoto-Daido type* models. Despite their simplicity, such systems provide a good framework to study emerging phenomena on networks of oscillators. The simplest of these properties, and yet ubiquitous in many real-world situations, is synchronization phenomena, which arises already when G is composed of a single harmonic.

1.3 A periodically forced oscillator

Some features regarding synchronization in large ensembles can be understood already from the analysis of a single oscillator driven by another unit. Let's assume a phase oscillator forced by a second, freely rotating, unit through a purely sinusoidal coupling function,

$$\begin{aligned}\dot{\phi} &= \omega + K \sin(\theta - \phi + \gamma) , \\ \dot{\theta} &= \Omega .\end{aligned}\tag{1.7}$$

Here $K \geq 0$ is the coupling strength, and $\gamma \in [0, 2\pi)$ is a parameter known as *Sakaguchi phase shift*. It is easier to study this system by analyzing the behavior of the phase difference between ϕ and θ . Introducing the change of variables $\psi = \phi - \theta$ the previous system can be written in the form

$$\dot{\psi} = \nu + K \sin(\gamma - \psi).\tag{1.8}$$

where $\nu = \omega - \Omega$. System (1.7) is a one-dimensional autonomous ordinary differential equation, and thus, its analytical treatment is transparent. As illustrated in Figure 1.3(a), in the simple case where $\nu = 0$ and $\gamma = 0$, equation (1.8) has two fixed points, $\psi_0 = 0$ and $\psi_1 = \pi$. The first one is stable, and corresponds to the case where the two oscillators evolve exactly at the same phase, *i.e.*, they are synchronized. The second corresponds to an *antiphase* solution, where the oscillators evolve one opposed to the other one.

As indicated in Figure 1.3(a), increasing (resp. decreasing) the difference between the natural frequencies of each oscillator, ν , shifts the value of the stable fixed point to the right (resp. left) and the unstable fixed point to the left (resp. right). A solution $\psi \neq 0$ means that the oscillators become *entrained* or *phase locked*: they evolve at the same frequency, but with a difference on their phases. Finally, when $\nu = K$, the two fixed points collide and vanish through a *saddle node* bifurcation. Figure 1.3(b) shows the parameter space (ν, K) the region of stability of the phase-locked solution, $K < |\nu|$. The triangular shape of this stability region is known as *Arnold tongue*, and it indicates that, the larger is

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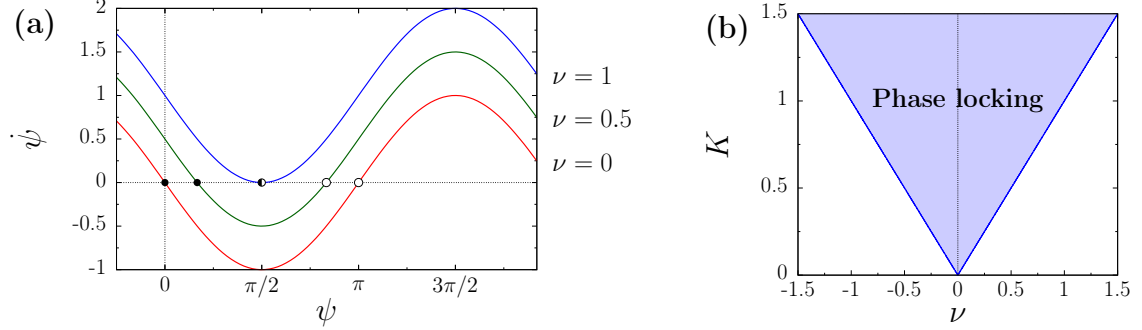


Figure 1.3: (a) Velocity field of the forced oscillator for different values of ν . (b) Arnold tongue (blue region).

the difference between natural frequencies, the larger needs to be the coupling strength K to allow for an entrained solution. Thus, if one of the oscillators is naturally much faster than the other one, it is not going to be possible to phase lock them unless accounting for very strong coupling. Nevertheless, there exist a number of other ways in which the oscillators can be entrained in this circumstance, for instance, 2 to 1, entrainment, which means that for 2 cycles of the first oscillator, the second runs exactly 1 period. For more details on these other solutions one can refer to [12].

Finally, the role of the Sakaguchi phase shift, γ , is that of modifying the way in which the two oscillators interact. We have seen that for $\gamma = 0$ and $\nu = 0$ the oscillators always synchronize. Keeping the natural frequencies identical, $\nu = 0$, and changing γ , the fixed points ψ_0 and ψ_1 shift to the right by γ , so that the oscillators might be entrained, but not evolving exactly at the same phase. In the limiting case of $\gamma = \pi$, the synchronized solution is unstable and the antiphase solution is stable, thus, the coupling is repelling.

1.4 The Kuramoto-Sakaguchi model

Having understood the concept of phase locking in a small system, we move now to collective synchronization. Let us consider now the Kuramoto-Sakaguchi model [20]: a system composed of N oscillators, where all the units are interacting with the rest of the network

1.4. The Kuramoto-Sakaguchi model

via a sinusoidal coupling function. This is, the phase of the j -th oscillator, $\phi_j \in [0, 2\pi)$, evolves according to

$$\dot{\phi}_j = \omega_j + \frac{K}{N} \sum_{m=1}^N \sin(\phi_m - \phi_j + \gamma) \quad j = 1, \dots, N ; \quad (1.9)$$

where ω_j are the natural frequencies of the different oscillators, $K > 0$ is the coupling strength, and $\gamma \in [0, 2\pi)$ is the Sakaguchi phase-shift. For $\gamma = 0$ one recovers the original Kuramoto model, which has been largely studied as a paradigm model for collective synchronization phenomena. The behavior of this system can be characterized by the *complex Kuramoto order parameter*

$$R = |R|e^{i\alpha} = \frac{1}{N} \sum_{j=1}^N e^{i\phi_j} \in \mathbb{C}, \quad (1.10)$$

which is nothing else than the *mean-field* of the system in the complex plane. If all the oscillators evolve closely together, then $|R| \simeq 1$, whereas if the oscillators evolve incoherently one to the other, then $|R| \simeq 0$, as represented in Figures 1.4(a,b). Moreover, the Kuramoto order parameter allows us to rewrite system (1.9) as

$$\dot{\phi}_j = \omega_j + K|R| \sin(\alpha - \phi_j + \gamma) . \quad (1.11)$$

Notice the similarity of this equation with that of a driven oscillator (1.7). There are, nevertheless two important differences. The first one is that the coupling term depends on a new parameter $|R|$, as if the forcing had also an amplitude. The second, and more crucial, is that the evolution of $|R|$ and α depends on the entire ensemble of oscillators, including the oscillator j itself. Hence, mathematically, the problem needs to be treated *self-consistently*, using the definition of the order parameter (1.10). Nevertheless, one can anticipate that concepts such as Arnold tongues and phase locking will also play an important role here.

We first study the original Kuramoto model, *i.e.*, with $\gamma = 0$. To illustrate the ef-

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fect of the coupling, Figure 1.9(c) shows the final value of $|R|$ for different values of the coupling strength, computed through simulations with $N = 1024$ oscillators (see blue circles and green triangles). The natural frequencies of the single oscillators are set randomly according to a Lorentzian (Cauchy) distribution, with scaling parameter Δ , $g(\omega) = [\pi\Delta(1 + \frac{x^2}{\Delta^2})]^{-1}$. For small coupling strength $|R|$ remains small, indicating that the system evolves incoherently. As K increases, there is a second-order phase transition towards synchrony. For $\Delta = 1$ the bifurcation occurs at $K_c \simeq 2$ (see blue circles), whereas if $\Delta = 2$ the distribution of natural frequencies is wider, and the transition occurs at $K_c \simeq 4$ (see green triangles). This is an example of Arnold tongues affecting collective synchronization.

The effect of the Sakaguchi phase shift is that of changing the role of the coupling function, going from attractive ($\gamma = 0$) to repulsive ($\gamma = \pi$). Thus, if $\pi/2 < \gamma < 3\pi/2$ the Kuramoto model does not display any transition to synchrony and the oscillators remain in an incoherent state. For $\gamma < \pi/2$ the bifurcation point is shifted to the left, as represented by the amber crosses in Figure 1.9.

On the limit of identical oscillators, $\omega_j = \omega \forall j$, there is no interplay between the natural frequency of each oscillator and the effect of the coupling: the Arnold tongue effect vanishes. Mathematically, this becomes evident eliminating the coupling strength parameter through a time rescaling, $t \rightarrow t/K$. In fact, one can also get rid of the natural frequencies by sitting ourselves in a rotating frame $\theta_j = \phi_j - \omega t$, so that the Kuramoto-Sakaguchi model (1.11) now reads

$$\dot{\theta}_j = |R| \sin(\alpha - \theta_j + \gamma) .$$

This system has only one parameter, γ : if $\gamma = 0$ all the oscillators evolve in perfect synchrony, and when $\gamma = \pi/2$ there is a discontinuous transition towards an incoherent regime. Despite of its simplicity, the homogeneous Kuramoto-Sakaguchi model establishes an important base point for many of the results that will be discussed throughout this thesis.

1.5. Macroscopic approach and low-dimensional description

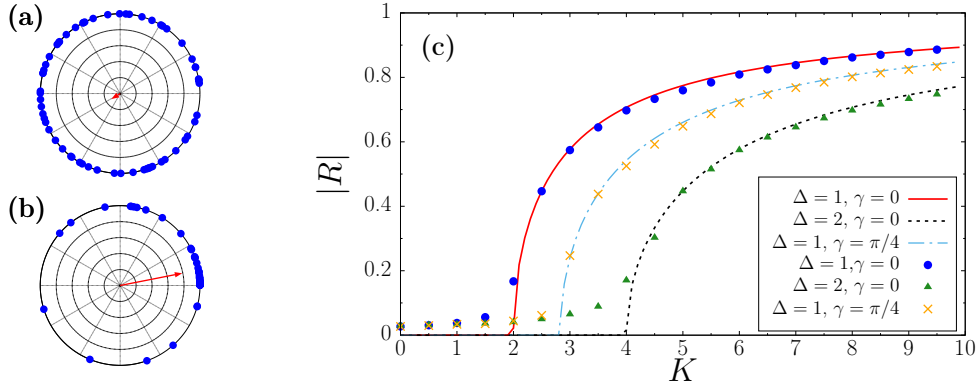


Figure 1.4: (a,b): Snapshot of the Kuramoto model in a disordered state (a), and an almost synchronous state (b). Blue circles represent the positions of the different oscillators in the unit circle. Red arrows represents the complex Kuramoto order parameter. (c) Modulus of the Kuramoto order parameter versus coupling strength in the Kuramoto-Sakaguchi model. Blue circles, green triangles, and amber crosses correspond to simulations of the microscopic system (1.9) with $N = 1024$ oscillators. Red continuous line, black dashed line, and cyan point-dashed line correspond to the results obtained using the Ott-Antonsen Ansatz. The natural frequencies of each oscillator are drawn from a Lorentzian distribution with scale parameter Δ . Blue circles and red line correspond to $\Delta = 1$ and $\gamma = 0$. Green triangles and black dashed line correspond to $\Delta = 2$ and $\gamma = 0$. Amber crosses and cyan point-dashed line correspond to $\Delta = 1$ and $\gamma = \pi/4$.

1.5 Macroscopic approach and low-dimensional description

In the limit of $N \rightarrow \infty$ one can describe the system macroscopically by means of a (nonlinear) Liouville operator. Let $P(\phi, \omega, t)$ be the density of oscillators with natural frequency ω that have phase ϕ at time t . Then, the mean-field (1.10) now reads

$$R = |R|e^{i\alpha} = \int_0^{2\pi} d\psi \int_{-\infty}^{\infty} d\omega P(\psi, \omega, t) e^{i\psi} \quad (1.12)$$

and the velocity field (1.11) becomes

$$\dot{\phi} = \omega + \frac{K}{2i} \left(R e^{-i(\phi-\gamma)} - R^* e^{i(\phi-\gamma)} \right) =: B(\phi, \omega, R) .$$

Chapter 1. Introduction to phase oscillators

Thus, using the conservation law of number of oscillators, the macroscopic description of system (1.9) is given by the following Liouville-type operator,

$$\frac{\partial}{\partial t} P(\phi, \omega, t) = -\frac{\partial}{\partial \phi} \{B(\phi, \omega, R) P(\phi, \omega, t)\} . \quad (1.13)$$

This expression is a nonlinear partial integro-differential equation that needs to be solved self-consistently with Eq. (1.12). Since P is a ϕ -periodic function, Fourier analysis provides a proper basis to study its properties under the corresponding evolution operator. Let us define the *Fourier modes* of P as

$$\tilde{P}(k, \omega, t) = \frac{1}{g(\omega)} \int_0^{2\pi} P(\phi, \omega, t) e^{ik\phi} d\phi .$$

Thus, the *Fourier transform* of P reads

$$P(\phi, \omega, t) = \frac{g(\omega)}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{P}(k, \omega, t) e^{-ik\phi} ,$$

Hence, determining the time-evolution of all the Fourier modes of P equals to solve the continuity equation (1.13). Notice that, since P is a real function, the relation $\tilde{P}(k, \omega, t) = \tilde{P}^*(-k, \omega, t)$ always holds. Ott and Antonsen showed in 2008 that it is possible to reduce such infinite dimensional problem to a system of two ODEs [21]. The so-called *Ott-Antonsen Ansatz* consists on relating all the Fourier modes through

$$\tilde{P}(k, \omega, t) = [\alpha(\omega, t)]^k .$$

Inserting the Fourier expansion of P in the macroscopic equation (1.13), and making use of the Ansatz, one finds that the evolution of $\alpha(\omega, t)$ is given by

$$\dot{\alpha}(\omega, t) = i\omega\alpha(\omega, t) + \frac{K}{2} [Re^{i\gamma} - R^* e^{-i\gamma} \alpha(\omega, t)^2] .$$

1.5. Macroscopic approach and low-dimensional description

Also, the self-consistent condition (1.12) now reads

$$R = \int_{-\infty}^{\infty} g(\omega) \alpha(\omega, t) d\omega . \quad (1.14)$$

Thus, given a initial condition $\alpha(\omega, 0)$ the dynamics of the Kuramoto-Sakaguchi system is ruled by a two-dimensional system (notice that α is a complex function). Moreover, for specific distributions of the natural frequencies $g(\omega)$, equation (1.14) can be solved explicitly. In particular, if $g(\omega)$ is Lorentzian with scaling parameter Δ , then the dynamics of the system can be described directly by the evolution of the complex Kuramoto order parameter,

$$\dot{R} = -\Delta R + \frac{K}{2} (R e^{i\gamma} - |R|^2 R^* e^{-i\gamma}) .$$

The red line, black dashed line and cyan point-dashed line in Figure 1.4(c) show the outcome of this latest equation for the values of Δ and γ studied microscopically.

The macroscopic solutions characterized by the Ott-Antonsen manifold $\tilde{P}(k, \omega, t) = \alpha(\omega, t)^n$ not only provide an effective low-dimensional description of the Kuramoto-Sakaguchi model: they are the only attracting solutions of the system in the thermodynamic limit provided that $g(\omega)$ is analytic. In the limiting case of identical oscillators, in principle, the Ott-Antonsen Ansatz is not globally attracting and one needs to rely on a more general theory, developed years before by Watanabe and Strogatz [13, 14]. According to this theory, the Kuramoto-Sakaguchi model is partially integrable, and it can be described by three global variables (one of them completely determined by the other two), and infinitely many constants of motion, being the Ott-Antonsen manifold a specific choice of such constants [16, 15]. Thus, although useful in many different relevant contexts, the behavior of ensembles of phase oscillators with sinusoidal coupling is relatively simple and quite well understood. Throughout this thesis we show how different types of coupling in networks of identical oscillators lead to richer dynamical regimes that escape the low-dimensional description provided by the Watanabe-Strogatz theorem.

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Chapter 2

A minimal model of self-consistent partial synchrony

2.1 Introduction

Adiabatic elimination of the amplitude through the phase reduction method provides an effective route to analyze collective behavior of systems composed by coupled oscillatory units in a generic way. Nevertheless, a comprehensive theory describing how phase oscillatory systems behave is so far missing. Indeed, such a theory would require to account for many features, such as the structure of the interaction between the single units, the type of coupling, and the level of heterogeneity among the different oscillators. The Kuramoto-Sakaguchi model and its low-dimensional description through the Watanabe-Strogatz theorem constitutes an important milestone. Despite its simplicity, it has been proved a useful model to describe many real-world phenomena, such as neural networks [1], power grids [2], and circadian rhythms [3]. On the other hand, the assumption of a sinusoidal coupling proves insufficient to reproduce more complex phenomena without accounting for nonlinear coupling, special network topologies, noise or time-delay.

Already on homogeneous systems coupled through mean-field, where the dynamics of each unit is identical to the rest, it is not clear what types of dynamical regimes one can ex-

Chapter 2. A minimal model of self-consistent partial synchrony

pect. Macroscopically one can classify the regimes emerging in this context by the shape of the probability density of the phases. On one side, there are clustered states, where the oscillators are spread in a collection of Dirac δ functions. On the other side one finds splay states, where the oscillators are spread uniformly. A mix of these two situations can be found in the famous *chimera states*, where the oscillators spontaneously spread between a fully synchronized cluster and a incoherent group through symmetry-breaking. Most of the efforts have been devoted to the study of clustered [4] and chimera states [5, 6] and much less to the identification and analysis of regimes characterized by a smooth but non-uniform distribution of phases. In some trivial cases this type of solutions are attainable in states where the oscillators are phase-locked on the macroscopic distribution [7, 8, 9]. However, is of special interest the situation where the frequency of the single units is incommensurate with that of the mean field. Such regimes, are herein referred to as self-consistent partial synchrony (SCPS). The simplest form SCPS consists of a periodic collective evolution in which the microscopic dynamics is quasiperiodic. Examples of such dynamics are: integrate-and-fire (IF) neurons interacting through finite-width pulses (the so-called α -functions) [10]; nonlinearly coupled Stuart-Landau systems [11]; a Kuramoto-like model obtained via phase reduction from the above mentioned Stuart-Landau ensemble. Also a variant of quasiperiodic partially synchronous dynamics has been detected in models beyond phase approximation, *i.e.*, in globally coupled Hindmarsh-Rose neurons and Stuart-Landau oscillators; where the macroscopic and microscopic frequencies are equal only on average, but the motion of oscillators is additionally modulated by a generally incommensurate frequency [12, 11].

In this chapter we further illustrate the richness of the Kuramoto-Daido model by showing that SCPS spontaneously emerges in a minimal extension of the Kuramoto-Sakaguchi setup to a biharmonic model, where the coupling function G is composed of just two harmonics. The simplicity of the biharmonic model allows for a detailed analysis of SCPS, which can be seen as a stationary solution of a continuity equation in a suitably rotating frame. Accordingly, the phase-distribution can be accurately determined from the emergence of SCPS out of the splay state – through a Hopf bifurcation – to its collapse onto full synchrony as in [13, 9]. The additional stability analysis confirms the numerical

evidence of the onset of an instability and allows identifying the unstable direction. The biharmonic model had been introduced already in previous works to study a switching dynamics between two cluster states [14, 15]. We revisit the stability properties of clusters, to understand under which conditions trajectories converge towards an heteroclinic cycle (HC). Moreover, we find that heteroclinic cycles can be viewed as a different type of self-consistent partial synchrony. Finally, to further explore the ubiquity of SCPS, we also study its emergence in an ensemble of linearly coupled Rayleigh oscillators. Performing the phase reduction method, we reconstruct the corresponding coupling function which turns out to contain a few harmonics. The resulting Kuramoto-Daido setup is shown to reproduce the dynamics of the original system.

This work has been done in collaboration with Michael Rosenblum (Potsdam University).

2.2 The biharmonic model

In this section we introduce the main reference model: an ensemble of identical, globally-coupled Kuramoto-Daido oscillators. The phase of each unit, $\phi_j \in [0, 2\pi)$, evolves according to

$$\dot{\phi}_j = \frac{1}{N} \sum_{m=1}^N G(\phi_m - \phi_j) \quad j = 1, \dots, N. \quad (2.1)$$

The coupling function $G(\phi)$ is assumed to have a biharmonic shape

$$G(\phi) := \sin(\phi + \gamma_1) + a \sin(2\phi + \gamma_2),$$

where γ_1 and γ_2 are Sakaguchi phase shifts, $\gamma_1, \gamma_2 \in [0, 2\pi)$, and the amplitude of the second harmonic is set to $a = 0.2$. The dynamics of the phase oscillators can be characterized

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with the help of the *Kuramoto-Daido order parameters*,

$$\tilde{P}(k, t) := \frac{1}{N} \sum_{j=1}^N e^{ik\phi_j} . \quad (2.2)$$

Such macroscopic observables contain all the necessary information to characterize the dynamics of the system for large N . The modulus of the first order parameter $|\tilde{P}(1, t)|$ correspond to the usual *Kuramoto order parameter* introduced in the previous chapter. It describes the level of synchronization of the network of oscillators, being equal to 1 when all the units evolve at the same phase, and equal to 0 when they are spread uniformly around the circle. On the other hand, the argument of $\tilde{P}(1, t)$ acts as a *global phase*

$$\Phi = \arg \left[\tilde{P}(1, t) \right] .$$

Its average growth rate determines the macroscopic frequency of the system, $\Omega = \langle \dot{\Phi} \rangle$, which in some cases is to be compared with that of the single units,

$$\nu = \langle \dot{\phi}_j \rangle .$$

The first two Kuramoto-Daido order parameters suffice to express the evolution equation (2.1) in the more compact form

$$\dot{\phi}_j = -\frac{i}{2} \left[\tilde{P}(1, t) e^{-i(\phi_j - \gamma_1)} + a \tilde{P}(2, t) e^{-i(2\phi_j - \gamma_2)} - \text{c.c.} \right] , \quad (2.3)$$

where c.c. stands for complex conjugate.

2.2.1 A macroscopic description

In the limit of $N \rightarrow \infty$ oscillators, the macroscopic dynamics of the model is effectively characterized by the *probability density* of the phases at time t , $P(\phi, t)$. At the thermody-

2.2. The biharmonic model

namic limit, the definition of the order parameters (2.2) reads

$$\tilde{P}(k, t) := \int_0^{2\pi} d\psi P(\psi, t) e^{ik\psi} . \quad (2.4)$$

Thus, the Kuramoto-Daido order parameters correspond, in fact, to the *Fourier modes* of the probability density. Therefore, at a fixed time t , the order parameters determine the shape of P as a linear combination of harmonic functions as

$$P(\phi, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{P}(k, t) e^{-ik\phi} .$$

The velocity field (2.3) defining the evolution of the oscillators evolving with phase ϕ under the action of $\tilde{P}(1, t)$ and $\tilde{P}(2, t)$ now reads

$$\begin{aligned} \dot{\phi} &= -\frac{i}{2} \left(\tilde{P}(1, t) e^{-i(\phi-\gamma_1)} + a \tilde{P}(2, t) e^{-i(2\phi-\gamma_2)} - \text{c.c.} \right) \\ &= \int_0^{2\pi} d\psi G(\psi - \phi) P(\psi, t) \\ &=: B[\phi, \tilde{P}(1, t), \tilde{P}(2, t)] \end{aligned}$$

and, therefore, the evolution of the probability density is characterized by the continuity equation

$$\frac{\partial}{\partial t} P(\phi, t) = -\frac{\partial}{\partial \phi} \left\{ B[\phi, \tilde{P}(1, t), \tilde{P}(2, t)] P(\phi, t) \right\} . \quad (2.5)$$

Such nonlinear partial integro-differential equation is obtained as a conservation law of the number of oscillators, and is equivalent to the Liouville operator of a vector field, with the exception that the velocity flux B depends also on P through the two first Fourier modes defined by Eq. (2.4).

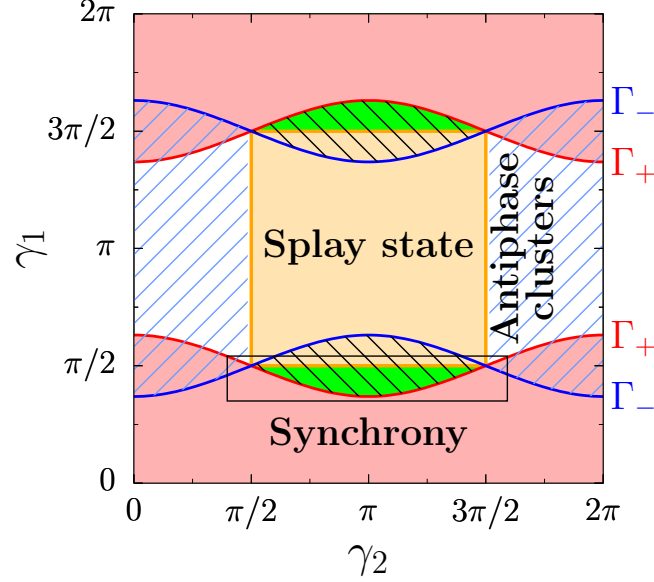


Figure 2.1: Phase diagram of the biharmonic model (2.1). The splay state is stable in all the amber central square region. The fully synchronous regime is stable in the top and bottom light red regions delimited the two branches of Γ_+ (blue). The antiphase two-cluster regime is stable in the left and right blue slantwise hatched region delimited by the two branches of Γ_- (blue) and the central square. The black slantwise hatched eyelet delimited by curves Γ_+ and Γ_- indicate the region where non-trivial two-cluster states exist and the inter-cluster exponent is negative. The green regions correspond to the regions where non of the simpler dynamical states are stable. The black box shows the range that has been explored numerically in figure 2.4.

2.3 Phase diagram

In this section we study with detail the shape and stability of the simpler dynamical regimes that emerge from system (2.1) upon tuning the parameters γ_1 , and γ_2 . Some of these fixed points can be studied already using the microscopic formulation of the model, whereas other rely on the macroscopic description. In both cases the stability is assessed by means of standard methods.

2.3.1 Full synchrony

The paramount solution of a homogeneous system of oscillators is the state of *full synchrony*. In such regime all the oscillators evolve with exactly the same phase $\phi_j = \phi_0$ $\forall j = 1, \dots, N$. Inserting such Ansatz in Eq. (2.1) one obtains,

$$\dot{\phi}_j = \frac{1}{N} \sum_{m=1}^N G(0) = G(0).$$

Therefore, the synchronized solution is not, in general, a stationary fixed point and the ensemble oscillates with frequency $G(0) = \sin(\gamma_1) + a \sin(\gamma_2)$. The macroscopic representation of this regime is a delta function $P(\phi, t) = \delta(\phi - \phi_0(t))$ (see Fig. 2.2(a)) and the modulus of all the order parameters is equal to 1, representing thus the maximum level of synchrony.

Stability analysis of such a solution is easy at the level of a microscopic description. Let us consider a small perturbation $\delta\phi_j$ of the phase of oscillator j . The linearization of equation (2.1) around $\phi_j = \phi_0$ provides the evolution of the generic perturbation $\delta\phi_j$ as

$$\delta\dot{\phi}_j = \frac{1}{N} \sum_{m=1}^N G'(0)(\delta\phi_m - \delta\phi_j) = -G'(0)\delta\phi_j + \frac{1}{N} \sum_{m=1}^N G'(0)\delta\phi_m \quad \forall j = 1, \dots, N.$$

The resulting linear system of N equations can be written in a matrix form,

$$\begin{pmatrix} \delta\dot{\phi}_1 \\ \delta\dot{\phi}_2 \\ \vdots \\ \delta\dot{\phi}_N \end{pmatrix} = G'(0) \begin{pmatrix} \frac{1-N}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \\ \frac{1}{N} & \frac{1-N}{N} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1-N}{N} \end{pmatrix} \begin{pmatrix} \delta\phi_1 \\ \delta\phi_2 \\ \vdots \\ \delta\phi_N \end{pmatrix}.$$

The previous matrix is circulant and symmetric, so its spectral properties are well established. It has one zero eigenvalue $\lambda_0 = 0$ corresponding to a uniform eigenvector, $\delta\phi_j = 1$

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for all j , corresponding to the marginal stability of a uniform phase shift of the entire distribution of oscillators. The other $N - 1$ eigenvalues are all $\lambda_1 = -1$, so that the synchronous regime is linearly stable if

$$-\cos(\gamma_1) - 2a \cos(\gamma_2) < 0 .$$

Figure 2.1 depicts this region of stability delimited by the two branches of the curve Γ_+ defined as $0 = \cos(\gamma_1) + 2a \cos(\gamma_2)$ (see red curve).

2.3.2 Incoherent states

The regime conceptually opposed to full synchrony is the *splay state*, a solution where the oscillators are spread uniformly around the unit circle, $P(\phi, t) = \frac{1}{2\pi}$ (see Fig. 2.2(b)). In this case all the order parameters are zero, so that the coupling term in equation (2.1) vanishes, and the oscillators stay in a fix position.

Since the coupling term only involves the two first harmonics of P , any probability density that satisfies the condition $\tilde{P}(1, t) = \tilde{P}(2, t) = 0$, but $\tilde{P}(k, t) \geq 0$ for $|k| > 2$, is also a fixed point of the system. Thus, there exist infinitely many solutions of Eq. (2.5) where the oscillators are locked in a non-homogeneous state. We refer to these regimes as to *incoherent states*¹.

Stability analysis of the splay state

The stability analysis of the *splay state* is transparent from the macroscopic description provided by Eq. (2.5). Let $P_0(\phi, t) = \frac{1}{2\pi}$ be the splay state solution, and let us consider an infinitesimal perturbation $u(\phi, t)$ of P_0 . The corresponding linearized equation is

$$\frac{\partial u}{\partial t} = -\frac{1}{2\pi} \frac{\partial}{\partial \phi} B[\phi, \tilde{u}(1, t), \tilde{u}(2, t)]$$

¹Notice that in other works, some of these regimes had been referred to as trivial forms of SCPS. Nevertheless, since they present phase-locking and, moreover, the coupling term vanishes, we preferred not to consider them as SCPS.

2.3. Phase diagram

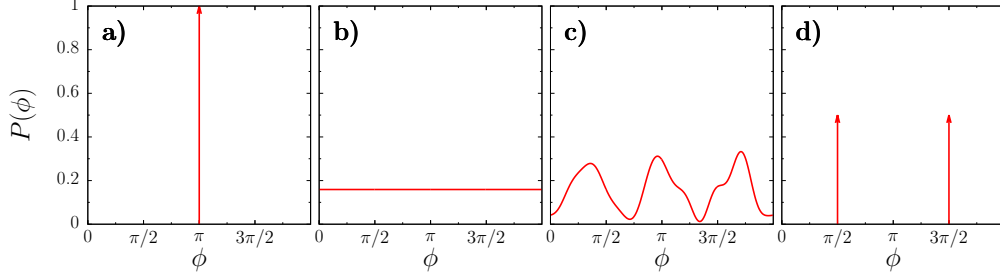


Figure 2.2: Probability density distribution $P(\phi)$ characterizing the simpler regimes of the biharmonic model. (a) Fully synchrony, where the height of the δ indicates its integral. (b) Splay state $P(\phi) = 1/(2\pi)$. (c) Incoherent state where $\tilde{P}(1) = \tilde{P}(2) = \tilde{P}(4) = 0$, $\tilde{P}(3) = 0.8$ and other modes are set randomly with smaller amplitudes. (d) Antiphase clusters.

where we have used that $B[\theta, \tilde{P}_0(1), \tilde{P}_0(2)] = 0$. Using the Fourier expansion of u one has that

$$\begin{aligned} \frac{\partial u}{\partial t}(\phi, t) &= \frac{1}{2\pi} \frac{\partial}{\partial t} \sum_{k=-\infty}^{+\infty} \tilde{u}(k, t) e^{-i\phi k} \\ &= \frac{1}{4\pi} \left(\tilde{u}(1, t) e^{-i(\phi - \gamma_1)} + 2a\tilde{u}(2, t) e^{-i(2\phi - \gamma_2)} + \text{c.c.} \right) . \end{aligned}$$

The evolution equations of the Fourier modes $\tilde{u}(k, t)$ are then given by

$$\begin{aligned} \frac{d}{dt} \tilde{u}(1, t) &= \frac{1}{2} e^{i\gamma_1} , \\ \frac{d}{dt} \tilde{u}(2, t) &= a e^{i\gamma_2} , \\ \frac{d}{dt} \tilde{u}(k, t) &= 0 \quad \text{if } |k| > 2 \end{aligned}$$

complemented by the complex conjugate equations for the negative k -modes. Manifestly, the equations are diagonal and the stability is determined only by the eigenvalues corresponding to the two first Fourier modes, $\lambda_1 = \frac{1}{2} e^{i\gamma_1}$ and $\lambda_2 = a e^{i\gamma_2}$. The stability boundaries of the splay state is then the central amber box in Figure 2.1. The directions corresponding to higher wavelength are marginally stable, corresponding thus to perturba-

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tions of the splay state that would lead to non-uniform homogeneous solutions.

Stability analysis of other incoherent solutions

Are there stable incoherent solutions in regions where the splay state is unstable? To answer this question one needs to assume a specific form of incoherent solution, and repeat the stability analysis performed in the previous subsection. Let's assume then that $P(\phi, t) = P(\phi)$ is a time-independent probability density with $\tilde{P}(1) = \tilde{P}(2) = 0$. Then it is still true that $B[\theta, \tilde{P}(1), \tilde{P}(2)] = 0$, and by linearizing the continuity equation one obtains that the evolution of the Fourier modes of an infinitesimal perturbation u is,

$$\begin{aligned} \frac{d}{dt} \tilde{u}(k, t) = & \frac{k}{2} \left(e^{i\gamma_1} \tilde{P}(k-1) \tilde{u}(1, t) + ae^{i\gamma_2} \tilde{P}(k-2) \tilde{u}(2, t) \right. \\ & \left. - e^{-i\gamma_1} \tilde{P}(k+1) \tilde{u}(-1, t) - ae^{-i\gamma_2} \tilde{P}(k+2) \tilde{u}(-2, t) \right), \end{aligned}$$

where $k = -\infty, \dots, \infty$. The corresponding infinite dimensional matrix has only four non-zero columns. If the j th column of the linear operator is zero, then the canonical vector $\mathbf{e}_j = \{\delta_{jm}\}_m$ is an eigenvector associated to a zero eigenvalue. Therefore, similarly than for the splay state, there are infinitely many marginally stable directions, and only four eigenvalues determining the stability of the solution. These four eigenvalues are given by $k = -2, -1, 1$, and 2 in the previous set of ODEs. The corresponding linear system is then

$$\begin{pmatrix} \dot{\tilde{u}}(-2) \\ \dot{\tilde{u}}(-1) \\ \dot{\tilde{u}}(1) \\ \dot{\tilde{u}}(2) \end{pmatrix} = \begin{pmatrix} ae^{-i\gamma_2} & 0 & -e^{i\gamma_1} \tilde{P}(-3) & -ae^{i\gamma_2} \tilde{P}(-4) \\ 0 & \frac{1}{2}e^{-i\gamma_1} & 0 & \frac{-a}{2}e^{i\gamma_2} \tilde{P}(-3) \\ \frac{-a}{2}e^{-i\gamma_2} \tilde{P}(3) & 0 & \frac{1}{2}e^{i\gamma_1} & 0 \\ -ae^{-i\gamma_2} \tilde{P}(4) & -e^{-i\gamma_1} \tilde{P}(3) & 0 & ae^{i\gamma_2} \end{pmatrix} \begin{pmatrix} \tilde{u}(-2) \\ \tilde{u}(-1) \\ \tilde{u}(1) \\ \tilde{u}(2) \end{pmatrix},$$

where we dropped the time dependence of $u(k, t)$ for simplicity. Notice that only the third and fourth Kuramoto order parameters are not associated with a marginally stable directions. For the particular case $\tilde{P}(3) = 0$, the real part of the resulting eigenvalues changes sign as those associated to the splay state. One can find general stability conditions for arbitrary $\tilde{P}(3)$ and $\tilde{P}(4)$ using, for instance, the Routh-Hurwitz criterion. Nevertheless, to prove that there is at least one incoherent state that is stable outside of the stability region of the splay-state it is enough to study the simpler case $\tilde{P}(4) = 0$, $\tilde{P}(3) = 1$. In this case, the previous matrix has two zero eigenvalues, and a pair of complex conjugate eigenvalues,

$$\lambda_{\pm} = e^{\mp i\gamma_1} + 2ae^{\pm i\gamma_2},$$

so that the corresponding stability region is exactly the complementary of the synchronous state one. Since we have not specified any restriction on the Fourier modes with wavelength $|k| > 4$, there are infinitely many solutions sharing this stability boundaries (see Figure 2.2(c) for a specific example). The most peculiar one is that where $\tilde{P}(k) = 1$ if k is multiple of 3, and zero otherwise. Such Fourier spectrum corresponds to a case where the oscillators are uniformly distributed in three equispaced groups or *clusters*, which brings us to the next class of solutions.

2.3.3 Clustered states and heteroclinic cycles

An *n-cluster state* consists of n families of oscillators where all the units within the same family evolve with identical phase. The fully synchronous regime represents the limiting case of one single cluster, and similarly to that case, a macroscopic representation of n clusters consists of a collection of n Dirac δ functions,

$$P(\phi, t) = \sum_{j=0}^{n-1} \alpha_j \delta(\phi - \phi_j(t)) ,$$

where ϕ_j is the phase that characterizes cluster j , and α_j is the proportion of oscillators at cluster j , $\alpha_0 + \dots + \alpha_{n-1} = 1$. If the clusters are of equal size and the corresponding

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phases ϕ_j are uniformly distributed between $[0, 2\pi)$, then $\tilde{P}(1, t) = 0$. Furthermore, if such uniform clustered state consists of more than two clusters, then $\tilde{P}(2, t)$ vanishes too, and the regime falls into the category of incoherent states. As showed in the previous section, in the biharmonic model only the case with $n = 3$ uniform clusters might be stable beyond the stability boundaries of the splay state.

Therefore, in the rest of this section we focus only on two-cluster states, Both for the sake of simplicity and since they are the most widely observed, here, we focus on the symmetric case of equal-size clusters. The resulting analysis can be easily extended to clusters with different sizes.

Two-cluster solutions

The phases ϕ_0 and ϕ_1 of the two families follow the differential equations

$$\dot{\phi}_0 = \frac{1}{2} [G(0) + G(\phi_1 - \phi_0)] \quad , \quad \dot{\phi}_1 = \frac{1}{2} [G(\phi_0 - \phi_1) + G(0)] \quad . \quad (2.6)$$

The two clusters are separated by a distance $\delta = \phi_0 - \phi_1$. Such separation variable evolves according to

$$\dot{\delta} = \frac{1}{2} [G(-\delta) - G(\delta)] = -G_A(\delta) \quad , \quad (2.7)$$

where G_A is the anti-symmetric component of the coupling function G ,

$$G_A(\delta) = \sin \delta (\cos \gamma_1 + 2a \cos \gamma_2 \cos \delta) \quad . \quad (2.8)$$

Two-cluster solutions are identified by fixed points of Eq. (2.7), *i.e.*, the zeros of G_A . There are various solutions of the equation $G_A(\delta) = 0$. The obvious one is $\delta_0 = 0$, which corresponds to the fully synchronous state. On the other hand, $\delta_0 = \pi$ corresponds to an *antiphase* two-cluster state. Two further solutions can be found by setting to zero the

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expression in parentheses in Eq. (2.8),

$$\cos \gamma_1 + 2a \cos \gamma_2 \cos \delta = 0 . \quad (2.9)$$

Such pair of solutions represent only one physically meaningful state: given a two-cluster state characterized by a separation δ_0 , the same state can be seen as characterized by a separation $2\pi - \delta_0$, if the two clusters are exchanged. Since $|\cos(\delta)| \leq 1$, solutions of Eq. (2.9) only exist in the parameter region delimited by

$$\Gamma_{\pm} := \cos \gamma_1 \pm 2a \cos \gamma_2 = 0 . \quad (2.10)$$

This equation with a plus sign corresponds to the case where δ reaches zero, and consistently defines the same curve as the stability condition for the synchronous state ($\delta = 0$). On the other hand, the equation with the minus sign corresponds to the case where the two cluster become antiphase ($\delta = \pi$). Therefore, the non-trivial cluster state ($0 < \delta < \pi$) is born from the antiphase clusters and dies out when the two clusters merge at the synchronous state.

Stability and heteroclinic cycles

The stability of a two-cluster state is determined by two different kinds of eigenvalues: the inter- and intra-cluster exponents. The inter-cluster exponent λ_I measures the stability against perturbation of the phase-separation between the two clusters. From Eq. (2.7) it follows

$$\lambda_I := -G'_A(\delta_0) = \cos \gamma_1 \cos \delta_0 + 2a \cos \gamma_2 \cos 2\delta_0 . \quad (2.11)$$

The intra-cluster exponents $\lambda_E^{\pm}$ measure the stability of the width of each cluster. In particular it determines the transversal stability of an oscillator that is forced by the coupling term without participating to it. From the equations of motion (2.6) it is readily found that

$$\lambda_E^{\pm} := -[G'_A(0) + G'_A(\delta_0) \pm G'_S(\delta_0)]$$

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where

$$G_S(\delta) := \frac{1}{2} [G(-\delta) + G(\delta)] = \sin \gamma_1 \cos \delta + a \sin \gamma_2 \cos 2\delta$$

is the symmetric component of G . The solution with the plus (resp. minus) sign refers to the cluster that is lagging behind (resp. leading) by δ_0 . The maximal intra-cluster eigenvalue is therefore,

$$\lambda_M := -\frac{1}{2} [G'_A(0) + G'_A(\delta_0) - |G'_S(\delta_0)|] . \quad (2.12)$$

Let's focus first to the antiphase cluster state. In this case the inter-cluster exponent $G'_A(\pi) = -\cos \gamma_1 + 2a \cos \gamma_2$, is negative in between the upper and lower branch of Γ_- and vanishes along Γ_- , confirming that the clustered solutions bifurcate out of the antiphase state. The stability of the antiphase state to intra-cluster perturbations is controlled by (see Eq. (2.12)),

$$\lambda_M = -2a \cos \gamma_2 , \quad (2.13)$$

so that this state is stable within the light blue hatched region in Fig. 2.1.

Let's move now to the nontrivial two-cluster state ($\delta_0 \neq \pi$). Whenever this solution exists (in the eyelets delimited by Γ_+ and Γ_- in Figure 2.1) the inter-cluster exponent has complementary stability boundaries to that of the fully synchronous and anti-phase clusters solutions because of the one-dimensional nature of the equation (2.7). In fact, Γ_+ indicates a pitchfork bifurcation, supercritical when $\pi/2 \leq \gamma_2 \leq 3\pi/2$ and subcritical otherwise. Therefore, the distance δ_0 between the two non-trivial clusters is only stable within the two black hatched eyelets in Figure 2.1.

By taking into account that $G_A(\delta_0) = 0$, one finds that

$$2\lambda_M = -\cos \gamma_1 (1 - \cos \delta_0) + |\sin \delta_0 (\sin \gamma_1 + 4a \sin \gamma_2 \cos \delta_0)| . \quad (2.14)$$

Therefore, such a two-cluster state is unstable for $\pi/2 < \gamma_1 < 3\pi/2$. A detailed analysis

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for $a = 0.2$ (with the help of a computer) reveals that the solution is unstable also within the eyelet e_1 .

As we show in Section 2.5, this is not yet the end of the story. The opposite sign of the two exponents $\lambda_E^\pm$ implies that, while the width of one cluster decreases, that of the other one diverges. However, as already discussed in [14], once the width of the “exploding” cluster becomes of order 1, nonlinear effects (not captured by a linear stability analysis) induce a relative phase shift of the two clusters, so that the leading cluster becomes the lagging one: this implies a stability “exchange”. As a consequence, the long term behavior can be assessed after averaging over the alternating periods of stability and instability. Symmetry reasons imply that the time duration of such two periods are equal to one another, so that the average exponent is, in general,

$$\lambda_a = -\frac{1}{2} [G'_A(0) + G'_A(\delta_0)] . \quad (2.15)$$

In the biharmonic model

$$\lambda_a = -\frac{1}{2} \cos \gamma_1 (1 - \cos \delta_0) .$$

In the region of interest, $\lambda_a < 0$ if $\cos \gamma_1 > 0$, *i.e.*, the fluctuations of the cluster widths on average decrease, without ever collapsing onto it. This is nothing but an attracting heteroclinic cycle (HC). A schematic representation of this situation is depicted in Figure 2.3.

2.4 Self-consistent partial synchrony

Most of the dynamical regimes studied so far share an important property: the evolution of the macroscopic observables (the Kuramoto-Daido order parameters) is identical to that of the single oscillators. For instance, the splay state and other incoherent states correspond to fixed points in a infinite dimensional space, and neither the Fourier modes of P nor the single oscillators evolve on time. On the other hand, at the fully synchronous state, both the macroscopic variables and the individual units evolve with the same constant frequency: there is *phase locking*. The exception to this claim is the heteroclinic cycle between two

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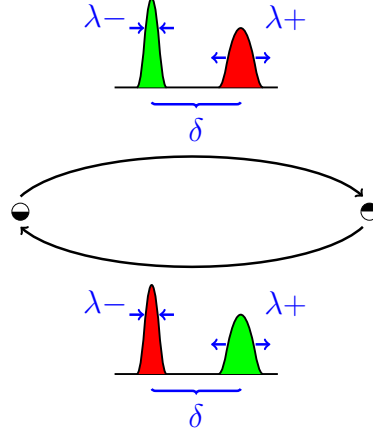


Figure 2.3: Schematic representation of the heteroclinic cycles between two two-cluster states. Each family of oscillators is depicted in a different color.

two-cluster state.

The novelty we present in this thesis, and the reason why we are studying the biharmonic model, is the existence of another stationary regime, namely, self-consistent partial synchrony (SCPS), where the macroscopic observables do not evolve in the same manner as the microscopic units. In other words, the single oscillators are not phase-locked on such distribution². In the biharmonic model SCPS arises in the simplest motion possible: the probability distribution P has a fixed smooth shape that evolve with a constant frequency, Ω . As a result, the macroscopic dynamics, identified by the generalized Kuramoto order parameters $\tilde{P}(k, t)$, is characterized by a one dimensional torus T^1 . Therefore, it can be studied analytically by sitting ourselves on a frame that rotates with the same velocity as P . This corresponds to rewrite $P(\theta, t) = P(\phi - \Omega t, t)$ in Eq. (2.5), so that it reads

$$\frac{\partial}{\partial t} P(\theta, t) = \frac{\partial}{\partial \theta} \left(\left\{ \Omega - B[\phi, \tilde{P}(1, t), \tilde{P}(2, t)] \right\} P(\theta, t) \right). \quad (2.16)$$

In this frame, the distribution of the oscillators at SCPS is nothing else than a fixed point

²As discussed in Section 2.5, according to this definition, HC can be interpreted also as a form of SCPS. Nevertheless, throughout this chapter we differentiate the two regimes to avoid confusion.

2.4. Self-consistent partial synchrony

on the previous equation, $P(\theta, t) \equiv P(\theta)$. Thus, the time derivative of P vanishes, which means that the phase derivative on the right hand side is actually zero and, upon integrating on the spatial variable, one obtains

$$-H = \left\{ \Omega - B[\phi, \tilde{P}(1), \tilde{P}(2)] \right\} P(\theta) , \quad (2.17)$$

where H is the constant of integration. We have put the minus sign on the right hand side of the previous equation so that H has the physical interpretation of *probability flux*. In other words, H it is the instantaneous proportion of oscillators that cross any point of the spatial domain $\phi \in [0, 2\pi)$. Therefore, $2\pi H + \Omega =: \nu$ corresponds to the average microscopic frequency at which the single oscillators evolve.

Now, solving Eq. (2.17) for $P(\theta)$ one obtains an implicit equation for the shape of the probability density,

$$P(\theta) = \frac{-H}{\Omega - B[\phi, \tilde{P}(1), \tilde{P}(2)]} . \quad (2.18)$$

In principle, this equation depends on six unknowns: the two complex variables $\tilde{P}(1)$ and $\tilde{P}(2)$, the macroscopic frequency Ω , and the probability flux H . Nevertheless, one is free to impose the phase of the probability distribution as $\arg[\tilde{P}(1)] = 0$. On the other hand, the probability flux H can be determined by imposing the normalization condition

$$\int_0^{2\pi} d\theta P(\theta) = 1 .$$

Then, equation (2.18) needs to be solved together with the self-consistent conditions

$$\tilde{P}(1) = \int_0^{2\pi} d\theta P(\theta) e^{i\theta} \quad \text{and} \quad \tilde{P}(2) = \int_0^{2\pi} d\theta P(\theta) e^{2i\theta} ,$$

which consists in finding a fixed point on a four-dimensional space. This problem can, thus, be solved by means of search methods, for instance, a genetic algorithm [16].

2.4.1 Stability analysis

In order to determine the stability of self-consistent partial synchrony, we consider an infinitesimal perturbation $u(\theta, t)$. Upon inserting $P(\theta) + u(\theta, t)$ in the continuity equation (2.16) and retaining first order terms one obtains that the evolution of u is given by

$$\frac{\partial}{\partial t} u(\theta, t) = \frac{\partial}{\partial \theta} \left(\left\{ \Omega - B[\theta, \tilde{P}(1), \tilde{P}(2)] \right\} u(\theta, t) - B[\theta, \tilde{u}(1, t), \tilde{u}(2, t)] P(\theta) \right).$$

The spectrum of this linear operator is purely point-like and can be determined by approximating the infinite-dimensional operator with finite matrices of increasing size. The most effective method consists in expanding $u(\theta, t)$ into Fourier modes

$$u(\theta, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{u}(k, t) e^{-ik\theta}.$$

As a result, the time evolution of u can be written as the time evolution of the different modes,

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{u}(k, t) = k \bigg\{ & -i\Omega \tilde{u}(k, t) + \frac{1}{2} e^{i\gamma_1} [\tilde{P}(1) \tilde{u}(k-1, t) + \tilde{P}(k-1) \tilde{u}(1, t)] \\ & - \frac{1}{2} e^{-i\gamma_1} [\tilde{P}(-1) \tilde{u}(k+1, t) + \tilde{P}(k+1) \tilde{u}(-1, t)] \\ & + \frac{1}{2} a e^{i\gamma_2} [\tilde{P}(2) \tilde{u}(k-2, t) + \tilde{P}(k-2) \tilde{u}(2, t)] \\ & - \frac{1}{2} a e^{-i\gamma_2} [\tilde{P}(-2) \tilde{u}(k+2, t) + \tilde{P}(k+2) \tilde{u}(-2, t)] \bigg\} \end{aligned}$$

where $k = -\infty, \dots, \infty$. Such infinite dimensional system of linear ordinary differential equations can be rewritten in the more compact form

$$\dot{\tilde{\mathbf{u}}} = \mathcal{S}(\tilde{\mathbf{P}}, \Omega) \tilde{\mathbf{u}} \tag{2.19}$$

where $\tilde{\mathbf{u}} = \{\tilde{u}(k, t)\}_{k=-\infty}^{\infty}$, $\tilde{\mathbf{P}} = \{\tilde{P}(k)\}_{k=-\infty}^{\infty}$, and the matrix $\mathcal{S}$ is defined as

$$\begin{aligned} \mathcal{S}_{mn}(\tilde{\mathbf{P}}, \Omega) := m \bigg\{ & -i\Omega\delta_{m,n} \\ & + \frac{1}{2}e^{i\gamma_1}[\tilde{P}(1)\delta_{m-1,n} + \tilde{P}(m-1)\delta_{1,n}] \\ & - \frac{1}{2}e^{-i\gamma_1}[\tilde{P}(-1)\delta_{m+1,n} + \tilde{P}(m+1)\delta_{-1,n}] \\ & + \frac{1}{2}ae^{i\gamma_2}[\tilde{P}(2)\delta_{m-2,n} + \tilde{P}(m-2)\delta_{2,n}] \\ & - \frac{1}{2}ae^{-i\gamma_2}[\tilde{P}(-2)\delta_{m+2,n} + \tilde{P}(m+2)\delta_{-2,n}] \bigg\}. \end{aligned} \quad (2.20)$$

Although this system is infinite dimensional, practically one selects a large but finite number of modes, $k = -M, \dots, M$, for the Fourier expansion of P and u and check *a posteriori* that progressively increasing M does not modify the results.

2.5 Results

2.5.1 Microscopic analysis

We explore the phase space using simulations of the microscopic equations (2.1) in the region indicated by the black rectangle in the bifurcation diagram of Figure 2.1. This area includes the region where synchronous, incoherent and two-cluster states are all unstable. Because of the system's symmetry mentioned above, the upper eyelet is characterized by equivalent dynamics.

Figure 2.4 summarizes the outcome of direct integration of the system (2.1) with $N = 1000$ oscillators initially distributed close to the splay state for a total time of 2×10^4 time units. Each circle corresponds to a simulation, and they are colored according to the different asymptotic regimes. The continuous curves correspond to the marginal stability lines of the phase diagram 2.1 discussed in the previous section. Overall, five different

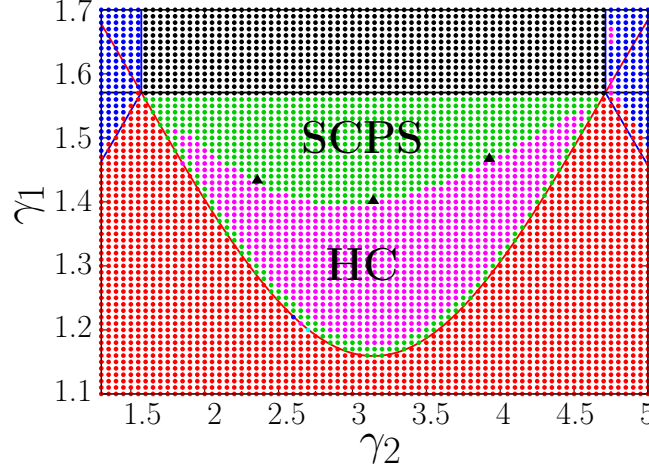


Figure 2.4: Map of dynamical regimes of the system (2.1), obtained via a direct numerical simulation, starting from a slightly perturbed splay state. The explored area corresponds to the black box in Fig. 2.1. The color of each circle indicates the observed regime: full synchrony in red, splay state in black, antiphase-clusters in blue, SCPS in green, heterocyclic cycles in pink, and three-cluster state in light blue (only appearing at $\gamma_2 = 2.7$, $\gamma_1 = 1.2$). The continuous curves are as in figure 2.1. The map has been computed for 1000 oscillators, with the transient time $2 \cdot 10^4$. The three black triangles correspond to points where SCPS is marginally stable as obtained from Eq. (2.19).

stability regions seem to be relevant. The splay state (black circles), fully synchrony (red circles), and antiphase clusters (blue circles) are, as expected, confined in the stability region determined analytically (compare with Fig. 2.1). In the bistable region between synchrony and anti-phase clusters, the trajectories converge to the antiphase cluster state. In the eyelet of interest, only two other dynamical regimes emerge: heteroclinic cycles (pink circles) and self-consistent partial-synchrony (green circles). Notice that the HC appears after a transient where the oscillators look distributed as SCPS. Such transient becomes larger upon approaching the transition curve to synchrony (red continuous curve in Figure 2.4). In fact, the belt of SCPS states surrounding the transition border in Fig. 2.4 appears to shrink upon increasing the computational time, so that it might not represent the asymptotic phase state. Before commenting more details about these two regimes, it is worth noticing that only one of the realizations has converged to a three-cluster state (see light blue circle at $\gamma_2 = 2.7$, $\gamma_1 = 1.2$).

Heteroclinic cycles

In computer simulations of identical oscillators, if two units end up with exactly the same phase, they will remain together for the entire simulation. In the HC state, oscillators belonging to the same cluster become closer after each cycle, so that, eventually, they separation becomes smaller than the available floating point representation, giving raise a spurious stable two cluster state. In order to avoid this effect, one needs to introduce a small heterogeneity in the ensemble. Namely, we set the natural frequencies of the oscillators, which are zero in the definition of the system (2.1), to be randomly distributed uniformly in the interval $[-0.5 \cdot 10^{-12}, 0.5 \cdot 10^{-12}]$. Whereas this change does not modify all the other states, it is crucial for the detection of HC.

The behavior of the HC is illustrated in Figure 2.5, corresponding to a simulation with $\gamma_1 = 1.35$, $\gamma_2 = \pi$ and 1000 oscillators. In Figure 2.5(a) we plot a section of the time evolution of the Kuramoto-order parameter, $\tilde{P}(1, t)$. The almost constant plateaus of the order parameter correspond to the intervals of time where the oscillators are distributed on almost two clusters, whereas the sharp drops indicate the stability switch. Figure 2.5(b) shows the time evolution of the phases corresponding to the single oscillators, ϕ_j (blue lines), and the phase of $\tilde{P}(1, t)$ (thick red line) in a frame co-rotating with the microscopic time average frequency, ν . The global phase (red line in Fig. 2.5(b)) grows quasi linearly on time. Its growth rate corresponds to that of the single oscillators when they are distributed on two clusters. Nevertheless, the corresponding average frequency, Ω , is clearly larger from that of the single oscillators, ν due to the permanent exchange of the leading and lagging clusters. Finally, it is worth noticing that during the different cycles, the two clusters do not exchange elements.

Phase transitions

In order to unveil the different transitions between asymptotic states, we perform a quantitative analysis along $\gamma_2 = \pi$. The time average of different order parameters is reported as black open circles in Figure 2.6(a-d), where three critical points can be identified:

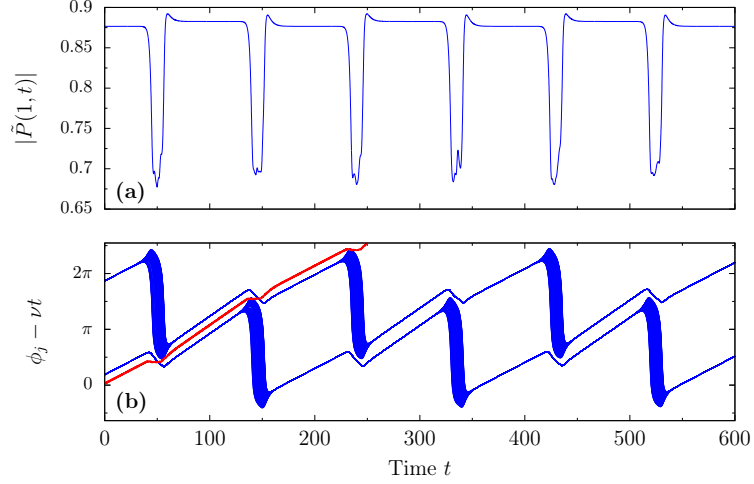


Figure 2.5: Dynamics of the heteroclinic cycle. (a) Time variation of $\tilde{P}(1, t)$ reflects the slow switching between cluster states. (b) Evolution of the phases of all oscillators (blue) and of the mean field (red) - after subtracting the average growth ν . For rather long epochs, oscillators are grouped into two clusters with almost identical phases. However, these states are unstable and alternately each group widens until its width becomes of order one, and then shrinks again. Notice, that there is no transfer of elements between the two groups, but the mean field frequency is larger than that of the single oscillators. Indeed, one can see that in the coordinate frame co-rotating with the microscopic frequency ν , the phase of $\tilde{P}(1, t)$ drifts away. Parameters are $\gamma_2 = \pi$, $\gamma_1 = 1.35$, and $N = 1000$. The initial conditions correspond to a perturbed splay state.

- (i) $\gamma_1^{(a)} = \pi/2$ signals the transition from splay state to SCPS;
- (ii) $\gamma_1^{(b)} \approx 1.401$ signals the transition from SCPS to HC.
- (iii) $\gamma_1^{(c)} = \arccos(2a) \approx 1.159$ signals the transition from HC to full synchrony.

As already illustrated by the analysis of section 2.3.2, at $\gamma_1^{(a)}$ the splay state loses stability through a pair of complex conjugate eigenvalues crossing the imaginary axis, and SCPS is born through a Hopf bifurcation. Thus, the order parameters $\langle |\tilde{P}(1, t)| \rangle$ and $\langle |\tilde{P}(2, t)| \rangle$ become positive, and the macroscopic oscillatory frequency Ω appears abruptly. After the transition, the average microscopic frequency of the single oscillators ν remains different and incommensurate to Ω , meaning that the single units are not phase-locked to the macroscopic distribution. The fact that ν remains smaller than Ω is not crucial for SCPS, since

the change of parameters $\phi \rightarrow -\phi$ reveals that on the upper eyelet of Figure 2.1 the same bifurcation scenario applies, but with opposite sign on the direction of the oscillations.

Further reducing γ_1 , SCPS loses its stability at $\gamma_1^{(b)}$, and the system falls into the HC regime. Whereas the transition is clear on the discontinuous jumps of $\langle |\tilde{P}(1, t)| \rangle$ and ν , the second Fourier mode, $\langle |\tilde{P}(2, t)| \rangle$, and the macroscopic frequency Ω do not show any symptom of a bifurcation. The switching between the stability of the two clusters makes the average microscopic frequencies differ from that of the mean-field. Finally, approaching $\gamma_1^{(c)}$, $\langle |P(1, t)| \rangle$ and $\langle |P(2, t)| \rangle$ become continuously closer to 1, as the synchronized regime becomes stable and the only rotation frequency becomes that of the synchronized regime. As commented before, close to the transition to synchrony the simulations seem to be dominated by increasingly large transients. Thus, the nature of such transition is not clear and will be discussed together with the stability of SCPS in the next section.

The abrupt jump in $\gamma_1^{(b)}$ indicates that SCPS loses stability through a subcritical bifurcation, so that there might a region of hysteresis. Figure 2.6(f-h) shows the outcome of simulations with adiabatically reducing (black circles) and increasing (orange line) γ_1 . Whereas the adiabatic decrease of the order parameter does not lead to new information (cf. black circles in Fig. 2.6(a-d)), slowly increasing γ_1 reveals that the stability boundaries of HC extend also to the region where SCPS is stable, thus both regimes share a zone of bistability.

2.5.2 Macroscopic analysis of SCPS

The macroscopic approach from section 2.4 allows us to explore the nature of SCPS and its stability properties. Red line in Figure 2.6 shows the results of solving the self-consistent equation (2.18) using a genetic algorithm³ [16]. The macroscopic observables clearly match the results obtained with microscopic simulations. Moreover, the existence of SCPS extends beyond its stability boundaries, providing a continuous link between the

³Simpler search algorithms such as Monte-Carlo methods can also be useful for this purpose.

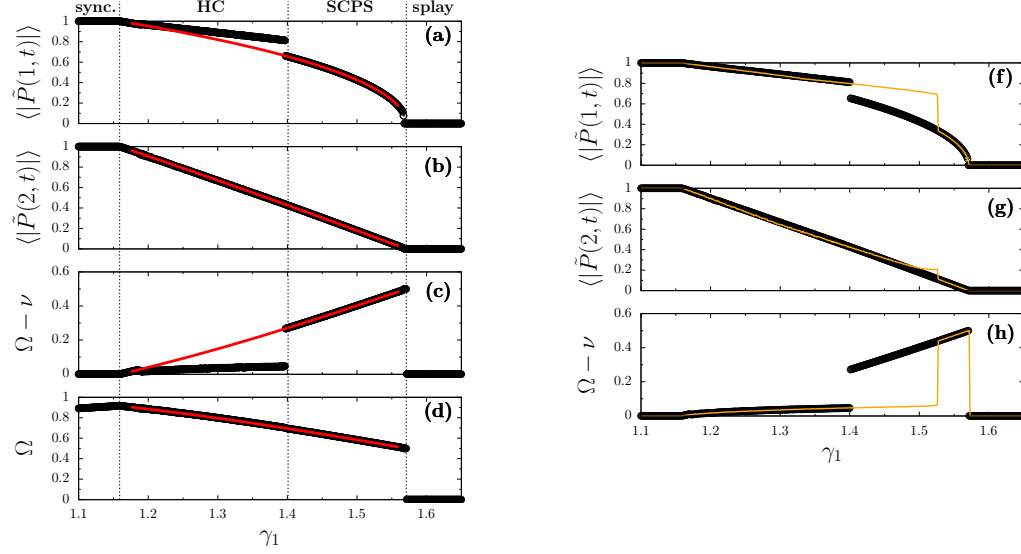


Figure 2.6: (a-d): Order parameters obtained from the microscopic simulations of Eq (2.1) (black open circles) and the macroscopic distribution of P given by Eq. (2.18) (red lines) with $\gamma_2 = \pi$. Microscopic simulations correspond to $N = 1000$ oscillators initially distributed close to the splay state. Time averages are computed over 10^4 time units after a transient of $5 \cdot 10^4$ time units. (a) First order Kuramoto parameter. (b) Second order Kuramoto parameter. (c) Difference between the time-average macroscopic frequency Ω and the time-average microscopic frequency ν . (d) Macroscopic frequency Ω . (a-c) Order parameters obtained from the microscopic simulations of Eq (2.1) with a slow decrease (black circles) and increase (orange lines) of γ_1 , with fixed $\gamma_2 = \pi$. Microscopic simulations correspond to $N = 1000$ oscillators. Time averages computed over 10^4 time units after a transient of 10^4 time units. (a) First order Kuramoto parameter. (b) Second order Kuramoto parameter. (c) Difference between the time-average macroscopic frequency Ω and the time-average microscopic frequency ν .

splay state and the fully synchronous solution. Indeed, the first and second Fourier modes of P go from 0 at $\gamma_1^{(a)}$ to 1 at $\gamma_1^{(c)}$, thus the shape of P is quite flat close to the splay state, and becomes sharper upon reducing γ_1 . For instance, Figure 2.7 shows the shape of $P(\theta)$ for $\gamma_2 = \pi$, and $\gamma_1 = 1.45$ (red line) and $\gamma_1 = 1.33$ (black dashed line).

The stability of SCPS is given by the eigenvalues of operator $\mathcal{S}$ defined in Eq.(2.20), which requires expanding P as a Fourier series up to some cut-off threshold M . Figures 2.8(a) and (b) show the resulting spectra for $\gamma_1 = 1.45$ and $\gamma_1 = 1.33$ respectively.

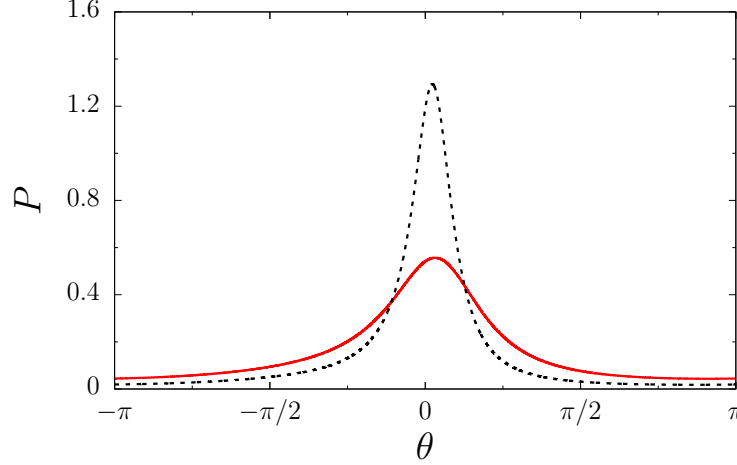


Figure 2.7: Probability density distribution of the oscillator's phases, obtained solving the self-consistent equation Eq. (2.18). Red continuous curve corresponds to $\gamma_1 = 1.45$, and black dashed line to $\gamma_1 = 1.33$.

The outcome of expansions with $M = 100$ (black crosses) and $M = 500$ (red circles) provide a good agreement for the relevant eigenvalues.

The stationary shape corresponding to SCPS has, at least two zero eigenvalues. One corresponds to the normalization condition of the probability density, as commented in section 2.4. The additional zero eigenvalue that appears in Figures 2.8(a,b) corresponds to a phase shift of the distribution. Indeed, we have verified that its associated eigenvector is proportional to the derivative of P . For $\gamma_1 = 1.45$, all the other eigenvalues are negative as expected for a stable state. Nevertheless, there is a large band very close to the imaginary axis, which seems to extend to infinite. Figure 2.8(c) shows that there is a clear exponential decrease of the real part (in absolute value) upon the imaginary part gets larger. Therefore, the tangent space associated to SCPS is folded with infinitely many weakly stable modes. This can be traced back on the microscopic simulations: the inset of Figure 2.8(c) shows the evolution of a uniform distribution initially confined to an interval of size $2\pi - \Delta_0$.

This is reminiscent of the stability of the incoherent states, which contain infinitely

many marginally stable solutions. Moreover, this shows that the biharmonic model breaks the existence of infinitely many conservation laws state by Watanabe-Strogatz theorem when the coupling function is purely sinusoidal. For $\gamma_1 = 1.33$ two pairs of complex conjugate appear to have positive real part, thus SCPS is no longer stable. In Figure 2.8(d) we keep track of the first pair of complex conjugates that cross the imaginary axis, and thus indicates the Hopf bifurcation. Using this procedure we identify the transition points marked with black triangles in Fig. 2.4. For values of γ_1 closer to the transition to synchrony, the shape of P gets too sharp to be identified properly. Nevertheless, from 2.8(d) it looks like SCPS keeps being unstable, and it vanishes just at the transition point, marked with a black dot at $\text{Re}(\lambda) = 0$.

Additional information can be obtained by looking at the eigenvectors. Unsurprisingly they are localized in the region where the probability density is concentrated. Moreover, the higher the imaginary part of an eigenvalue, the larger the number of oscillations of the corresponding eigenvector: this is a manifestation of the above mentioned (approximate) relationship between imaginary parts and wavenumbers. The eigenvectors corresponding to the unstable directions, $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are plotted in Fig. 2.9, separating the real from the imaginary component (see the solid lines). One can also see that $\mathbf{u}^{(1)}$ is also reminiscent of the first and second derivatives of P respectively (compare red continuous and black dashed lines in panels (a) and (b) in Fig. 2.9). The analogy extends to $\mathbf{u}^{(2)}$, if compared with the third and fourth derivatives of P , although it becomes much more qualitative (see panels (c) and (d) in Fig. 2.9).

2.6 SCPS in Rayleigh oscillators

In order to provide further evidence of the ubiquity of SCPS, we present a numerical study of globally coupled identical Rayleigh oscillators, already defined in the introduction. The equations are

$$\ddot{x}_j - \zeta(1 - \dot{x}_j^2)\dot{x}_j + \omega^2 x_j = \varepsilon \text{Re} [e^{i\gamma}(X + iY)] , \quad (2.21)$$

2.6. SCPS in Rayleigh oscillators

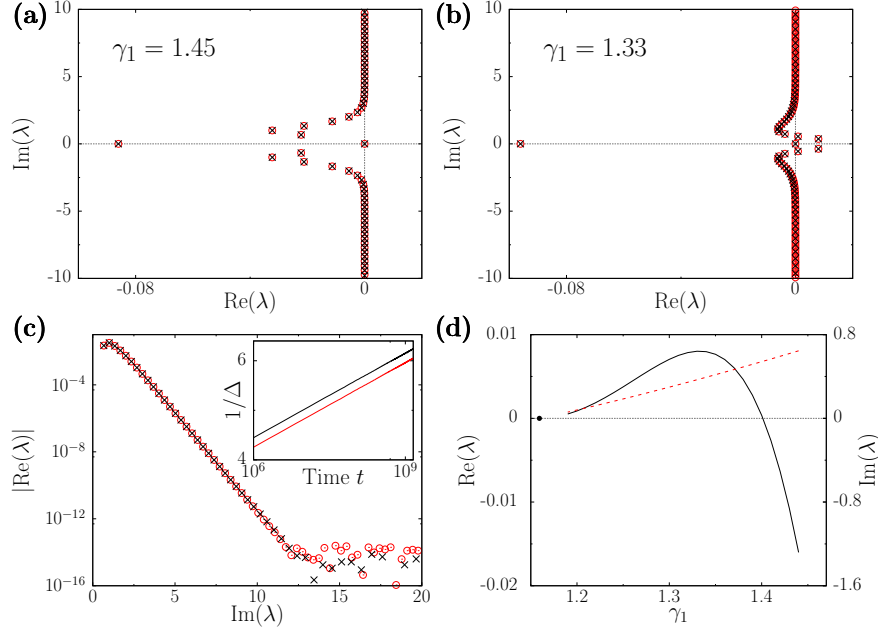


Figure 2.8: Stability analysis of SCPS obtained by diagonalizing $\mathcal{S}$ as given by Eq. 2.20). (a) Eigenvalues of the linear operator $\mathcal{S}$ for $\gamma_1 = 1.45$ and $\gamma_2 = \pi$ using a Fourier series of $M = 100$ (black crosses) and $M = 500$ (red circles) modes. (b) Same for $\gamma_1 = 1.33$. (c) Exponential decay of almost marginally stable eigenvalues of (a). Inset displays the evolution of an initially flat distribution with a gap $\Delta(0)$ for $\gamma_1 = 1.5$ and $\gamma_2 = \pi$, where SCPS is stable. The inverse gap-width $1/\Delta$ is shown as a function of time, for $\Delta(0) = \pi/7$ and $\pi/5$, (black and red lines, respectively). (d) Real (black line) and imaginary (red dashed line) parts of the pair of eigenvalues that first cross the imaginary axis. The results are shown only for the eigenvalue with positive imaginary part.

where $X = N^{-1} \sum_k x_k$ and $Y = N^{-1} \sum_k \dot{x}_k$ are two mean fields, while ε is the coupling strength. Finally, the control parameter γ accounts for a phase shift of the coupling term: it determines whether the interaction is attractive or repulsive in a similar way to the Sakaguchi phase shift on the Kuramoto-Daido models.

By introducing $y = \dot{x}$ the model reads

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \zeta(1 - y^2)y - x + \varepsilon \text{Re} \left[e^{i\gamma} (X + iY) \right]. \end{aligned}$$

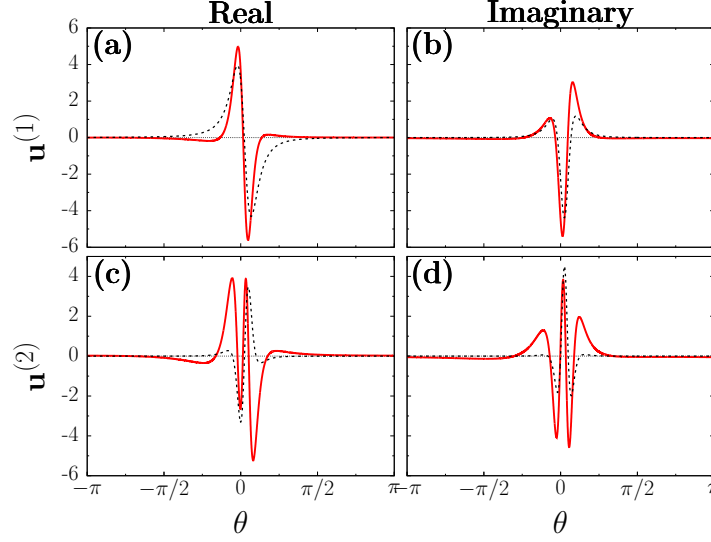


Figure 2.9: Eigenfunctions associated to the eigenvalues controlling the stability of SCPS for $\gamma = 1.33$. Panels (a) and (b) show real and imaginary part of the eigenfunction associated to $\lambda^{(1)}$ (red lines), and first and second derivatives of P suitable rescaled (black dashed lines) respectively. Panels (c) and (d) show real and imaginary part of the eigenfunction associated to $\lambda^{(2)}$ (red lines), and third and fourth derivatives of P suitable rescaled (black dashed lines), respectively.

As shown in the previous chapter, the uncoupled units in Eqs. (2.21) exhibit limit-cycle oscillations, while the nonlinearity parameter ζ determines the stability of the limit cycle. For $\zeta = 5$ the transversal Lyapunov exponent is -7.358, thus, adiabatic elimination of the amplitude is rather meaningful.

An appropriate order parameter is

$$\rho = \text{rms}(X)/\text{rms}(x) ,$$

where “rms” means root-mean-square of the time evolution. The splay state is characterized by a constant mean field and thereby $\rho = 0$. In the fully synchronous state, for identical oscillators, the microscopic and macroscopic dynamics are equivalent to one another so that $\rho = 1$.

2.6. SCPS in Rayleigh oscillators

The scenario resulting for $N = 1000$, $\varepsilon = 0.05$, $\zeta = 5$, and transient time $7.5 \cdot 10^5$ is reported in Fig. 2.10(a). It is reminiscent of that observed in the biharmonic model, with some differences. SCPS establishes itself in between the parameter range where full synchrony is stable (below $\gamma \approx -0.6$) and the region where ρ is negligible (above $\gamma \approx 0.05$) (see a snapshot in Figure 2.10(c3)). In this case, the mean field is slower than the individual oscillators, as in the upper eyelet of the biharmonic model, *cf.* Fig. 2.1. The regime characterized by a vanishing ρ is not asynchronous but a symmetric 9-cluster state (see a snapshot in Figure 2.10(c4)). In fact, the splay state turns out to be unstable in the full range of γ values that we have explored. Finally, in the interval $[-0.6, -0.2]$ one can see a slow convergence towards a two-cluster state (the large value of the order parameter ρ is due to the closeness between the two clusters, see Fig. 2.10(c2)).

The dynamics of the coupled system Eq. (2.21) can be better understood by performing a phase reduction. This can be done by introducing a phase variable ϕ for each individual oscillator, making reference to the uncoupled limit (*i.e.*, $\varepsilon = 0$) as illustrated in the introduction. Then, original Rayleigh oscillators (2.21) can be mapped onto a Winfree model 1.5

$$\dot{\phi}_k = \omega_0 + \frac{\varepsilon}{N} Z(\phi_k) \sum_j p(\phi_j) ,$$

where $Z(\phi)$ is the phase response curve (PRC), and $p(\phi)$ is the forcing function. The PRC $Z(\phi)$ can be obtained by following the standard approach: it corresponds to the phase shift imposed by an infinitesimal kick when the phase of the oscillator is ϕ . From the definition of the model (see Eq. (2.21)) it is clear that the direction of the external forces is on the y direction so that, following the notation of the introduction, $\delta \mathbf{x} = (0, 1)$. The resulting PRC is reported in Fig. 2.10(b) (see red curve). The forcing function p is instead obtained by expressing the coupling term due to the j th oscillator $\text{Re}[e^{i\gamma}(x_j + i\dot{x}_j)]$ as a function of ϕ_j ,

$$p(\phi) = x \left(\phi \frac{T}{2\pi} \right) \cos(\gamma_1) - y \left(\phi \frac{T}{2\pi} \right) \sin(\gamma) .$$

Its shape is depicted in Figure 2.10 (see black curve).

Chapter 2. A minimal model of self-consistent partial synchrony

Finally, as explained in the previous chapter, the coupling function of Kuramoto-Daido model can be obtained by computing the convolution of Z and p ,

$$G(\phi) := \int_0^{2\pi} d\psi Z(\psi - \phi) p(\psi) .$$

The resulting coupling function corresponds to the blue curve in Fig. 2.10(b). Fourier analysis shows that even modes are absent: this follows from the symmetry of the limit cycle: adding π to the phase results in changing the sign of both the PRC and forcing term.

In order to test the validity of the Kuramoto-Daido reduction, it has been simulated for eight harmonics. The time-averaged value of the Kuramoto order parameter is reported in black crosses in Figure 2.10(a). The resulting scenario is in close agreement with that exhibited by the original ensemble of Rayleigh oscillators (*cf.* with red circles). A series snapshots corresponding to different values of γ further validate the phase reduction method (see Figure 2.10(row c)). Within the Kuramoto-Daido representation, one can easily perform a stability analysis of both the splay and synchronous state. A direct extension of the stability of the splay state for the biharmonic model to a general coupling function with M harmonics shows that, also in this case, the stability of the splay state is determined by the imaginary components of the Fourier modes of G . In table 2.1 we show the contribution of the first eight nonvanishing components for $\gamma = 0.2$, where a zero order parameter is observed. In fact, the splay state turns out to be unstable because of the contribution of the 7th and higher harmonics.

As for the fully synchronous state, its stability is determined by the sign of $G'(0)$. The bifurcation point where it changes stability is $\gamma \simeq -0.573$ in agreement with the numerical simulations of the original model (2.21). For larger values of γ and below -0.2, the Kuramoto-Daido model possesses two-cluster states. Following the analysis outlined in Sec. 2.3.3, we conclude that two-cluster states are effectively stable. In fact, for $\gamma = -0.4$, the inter-cluster exponent $\lambda_I = -G'_A(\delta_0) = -0.0669$ is negative, while $\lambda_E^+ = 0.0869$, and

2.6. SCPS in Rayleigh oscillators

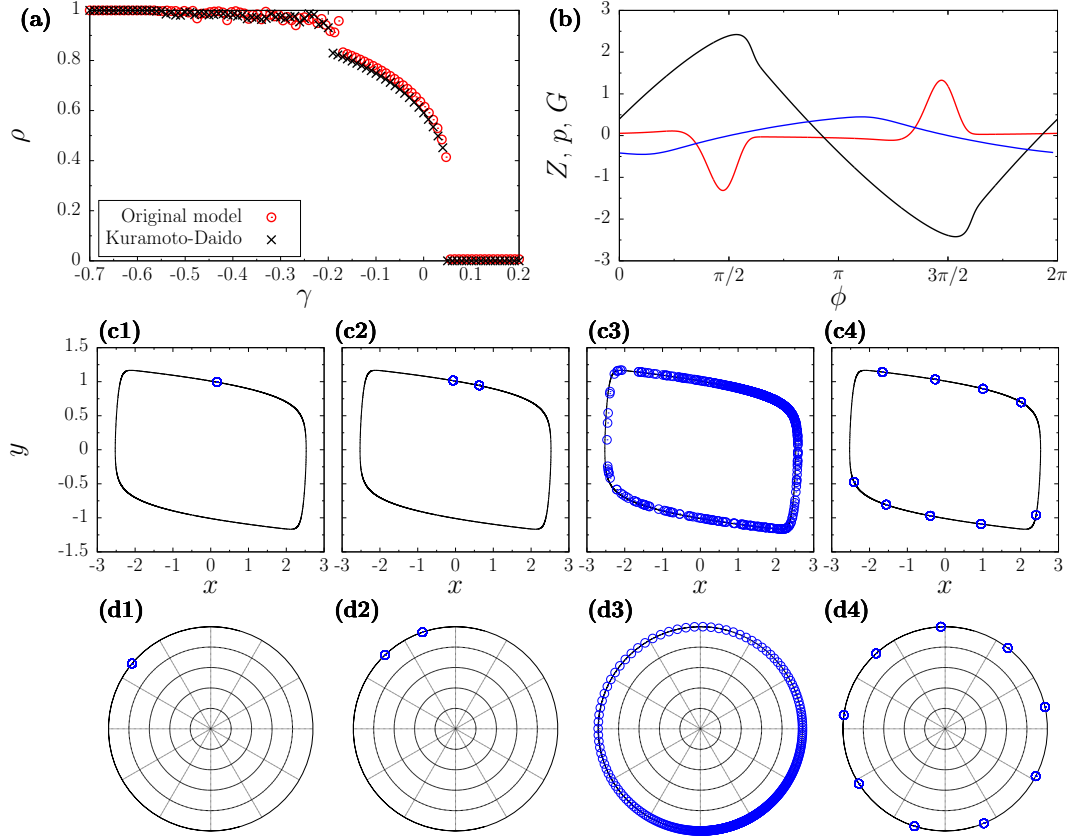


Figure 2.10: (a) Order parameter of the Rayleigh model, ρ (red circles), and that of the Kuramoto-Daido reduction, $\langle |\tilde{P}(1, t)| \rangle$ (black crosses), for different values of γ . From right to left, the system displays 9 clusters state, SCPS, 2 cluster state, and full synchrony. (b) Phase response curve Z (red curve), forcing function p (black curve), and the coupling function G (blue curve) of the phase reduction for $\gamma = 0.4$. Rows (a) and (b): snapshots of the Rayleigh system (a) and the Kuramoto-Daido reduction (b) for $\gamma = -0.6$ (full synchrony) (a1,b1); $\gamma = -0.4$ (2 cluster state) (a2,b2); $\gamma = 0.1$ (SCPS) (a3,b3); and $\gamma = 0.1$ (9 cluster state) (a4,b4).

$\lambda_E^- = -0.1146$. Accordingly the average exponent $\lambda_a = -0.01385$ is negative, in spite of one of the two intra-cluster exponents being positive. Altogether, the scenario is similar to that observed in the biharmonic model.

Index	Real part	Imaginary part
1	-1.6×10^{-2}	2.1×10^{-1}
3	-5.6×10^{-2}	1.5×10^{-2}
5	-1.6×10^{-2}	1.9×10^{-2}
7	3.0×10^{-3}	1.0×10^{-2}
9	4.0×10^{-3}	1.7×10^{-3}
11	1.5×10^{-3}	5.0×10^{-4}
13	3.8×10^{-4}	4.3×10^{-4}
15	7.9×10^{-5}	1.8×10^{-4}

Table 2.1: Eigenvalues associated to the stability of the splay state for $\gamma = 0.2$. The index refers to the Fourier mode which they are associated with.

2.7 Conclusions

In this chapter we have introduced and analyzed a Kuramoto-Daido model with a biharmonic coupling function. The addition of the second harmonic provides much richer dynamics than the usual Kuramoto-Sakaguchi model. In fact, it allows for two asymptotic regimes with a non-zero flux, *i.e.*, the frequency of the microscopic units differs from that of the total ensemble. The difference between those two dynamical regimes lies on the fact that HC is a non-stationary regime where the probability density evolves in time. On the other hand, periodic SCPS can be interpreted as a fixed point of the macroscopic distribution of the phases in a suitable rotating-frame. Whereas HC had been already studied in this very same model, this work is the first to analyze SCPS on the biharmonic model.

The simplicity of the model allows for an analytical approach to SCPS that unveils its stability properties. It is thus clear, that SCPS is born from a supercritical Hopf-bifurcation of the splay state, and it dies out through a subcritical transition. Nevertheless, the existence of SCPS covers the entire gap between the stability border of the splay state and full synchrony, thus naturally linking such both states. Moreover, the stability analysis of SCPS shows that a second harmonic suffices to break the conservation-laws of the Watanabe-Strogatz theorem.

Whereas SCPS can emerge in many different setups, this work proves that a second harmonic in the harmonic function suffices for it to emerge. It is important to notice that the Sakaguchi phase shift parameters γ_1 and γ_2 , play a major role, since they provide the non-symmetric shape of the coupling function that finally allows the emergence of SCPS. Nevertheless, the question of whether a specific coupling function can lead to even more complicated dynamics remains open.

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Chapter 3

Noise-induced stabilization of collective dynamics

3.1 Introduction

Typically, noise decreases the coherence of a dynamical system, by blurring, for instance, a perfect periodicity, or smoothing the fractal structure of low-dimensional chaos. Furthermore, noise can destabilize an attractor, when sufficiently large fluctuations allow overcoming an effective energy barrier. Such effects naturally occur whenever the unavoidable presence of stochastic forces is included into an otherwise deterministic evolution. Sometimes, however, noise may unexpectedly have the opposite effect of either increasing the overall coherence or stabilizing a given dynamical regime. Well known examples are the stabilization of the inverted pendulum [1], stochastic resonance [2, 3], and coherence resonance [4, 5].

In this chapter we discuss another such instance, where a finite but small amount of white noise, acting independently on each oscillator of the biharmonic model, stabilizes self-consistent partial synchrony (SCPS). This phenomenon differs from standard noise-induced bifurcations (see, for instance [6, 7]), where the Lyapunov exponent measuring the local stability of a given regime changes sign because of the fluctuations induced by

Chapter 3. Noise-induced stabilization of collective dynamics

a (multiplicative) stochastic force. Here, the noise acts on the microscopic level, while the regime we are interested in is a collective state. Macroscopically, SCPS corresponds to a rotation of the probability density of oscillator phases. Its dynamics, controlled by a nonlinear continuity (Liouville-type) equation, take place within an infinite-dimensional functional space. As showed in the previous chapter, one pair of eigenvalues of the linearized equations may have a positive real part, implying that SCPS is unstable and cannot be thereby maintained indefinitely. On the macroscopic level, the effect of noise is described by a diffusion operator, which adds up to the continuity equation, transforming it into a nonlinear Fokker-Planck equation. The impact on the overall dynamics may be relatively trivial, as the regularization of the heteroclinic cycle (HC) towards the switching dynamics studied in [8], but also much less so, when it stabilizes SCPS, as we show in this chapter.

The addition of noise to the biharmonic setup has also been studied in Ref. [9] for parameter values where, however, no “shear” phenomena such as SCPS can arise. Here, we explore instead a region where neither the fully synchronous, nor the splay state are stable in the deterministic limit. Numerical simulations of the microscopic equations performed for different noise levels show that a noise amplitude of 1.2×10^{-4} (see the next section for a proper definition) stabilizes SCPS. For smaller noise, a second regime is present: it is characterized by a pulsating bimodal distribution of phases, which can be traced back to the existence of an attracting heteroclinic cycle of cluster states. The transition between the two regimes is controlled by a bifurcation that can be either super or subcritical, this meaning that there exists a bistability region where, depending on the initial conditions, either of the two regimes can be attained. The details of the bifurcation diagram have been unraveled with the help of a perturbative approach, the noise amplitude being the smallness parameter. As a result, we have been able to estimate the deviations induced by the noise on the actual shape of the probability density of the phases, and then to solve the corresponding linearized equations in order to determine the stability properties of SCPS. These detailed studies of the biharmonic model are accompanied by a similar (but purely numerical) investigation of an ensemble of weakly coupled Rayleigh oscillators, where an

analogous stabilization of SCPS emerges when a small noise is added to the microscopic evolution equations.

3.2 The biharmonic model with noise

Let us recover the biharmonic model, but now with a stochastic additive force,

$$\dot{\phi}_j = \frac{1}{N} \sum_{m=1}^N G(\phi_m - \phi_j) + \xi_j(t) \quad j = 1, \dots, N ; \quad (3.1)$$

where $\phi_j \in [0, 2\pi)$ is the phase of the j -th oscillator, and $\xi_j(t)$ is a zero-average white noise such that $\langle \xi_j(t) \xi_m(t') \rangle = 2D \delta_{jm} \delta(t - t')$. Throughout this chapter we denote the generalized Kuramoto-Daido order parameters with a subindex referring to the level of noise in the system,

$$\tilde{P}_D(k, t) := \frac{1}{N} \sum_{j=1}^N e^{ik\phi_j} \xrightarrow{N \rightarrow +\infty} \int_0^{2\pi} d\phi P(\phi, t) e^{ik\phi} .$$

Recalling that the shape of the coupling function is

$$G(\phi) := \sin(\phi + \gamma_1) + a \sin(2\phi + \gamma_2)$$

where $\gamma_1, \gamma_2 \in [0, 2\pi)$, and $a = 0.2$, the compact form of Eq. (3.1) is now written as

$$\dot{\phi}_j = -\frac{i}{2} \left[\tilde{P}_D(1, t) e^{-i(\phi_j - \gamma_1)} + a \tilde{P}_D(2, t) e^{-i(2\phi_j - \gamma_2)} - \text{c.c.} \right] + \xi_j(t) . \quad (3.2)$$

We also add the subindex D on the averaged frequency of the single oscillators,

$$\nu_D := \langle \dot{\phi}_j \rangle ,$$

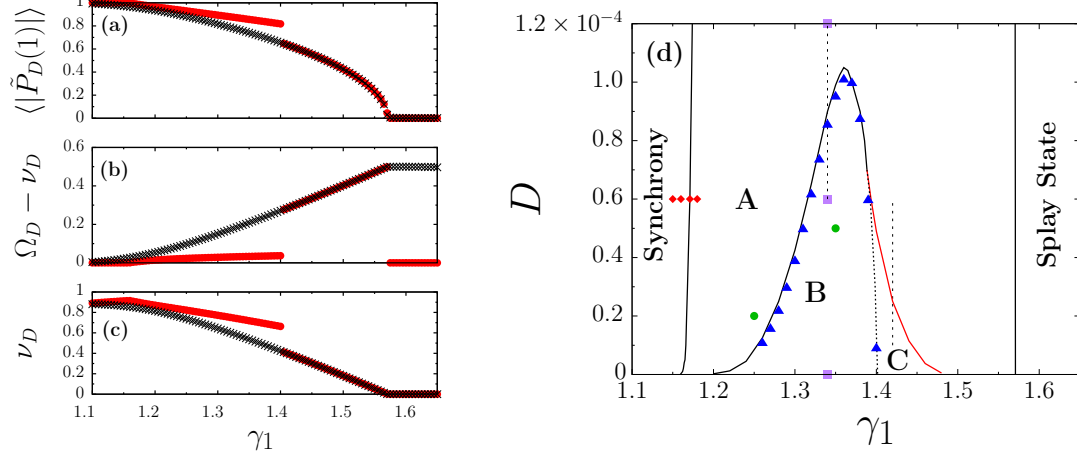


Figure 3.1: (a-c): Different order parameters versus γ_1 with fixed $\gamma_2 = \pi$, obtained from direct simulations with no noise (red circles), and $D = 1.2 \cdot 10^{-4}$ (black crosses). (a) Time averaged Kuramoto order parameter; (b) Difference between macroscopic and microscopic frequencies; (c) Microscopic frequencies of the system, obtained by averaging over the single frequencies of all the oscillators. Simulations correspond to the integration of Eq. (3.1) for $N = 1000$ oscillators. Each order parameter has been computed for 10^5 time units after discarding a transient of 10^4 time units. (d): Phase diagram in the plane (γ_1, D) . In region A only SCPS is stable, B indicates where only SD is stable, and C is a region of bistability. Black continuous and black dashed lines separating regions A, B, and C correspond to super and subcritical Hopf bifurcations respectively, whereas the red continuous line separating regions B and C indicates a saddle-node of periodic orbits bifurcation. These bifurcation lines have been obtained from direct simulations (see main text). Blue triangles are the results from the macroscopic description developed in Section 3.4. The two green circles indicate the points corresponding to simulations in Figure 3.2(a,b) and the three purple squares correspond to the simulations of Figure 3.2(c). Vertical black dashed lines indicate the range corresponding to Figure 3.3. The four red diamonds show the parameter values of the potentials in Figure 3.4.

and that of the ensemble,

$$\Omega_D := \langle \dot{\Phi} \rangle \quad \text{where} \quad \Phi = \arg \left[\tilde{P}_D(1) \right] .$$

3.3 Microscopic approach

In the previous chapter we have studied in detail a set of bifurcations that appear upon reducing γ_1 and with fixed $\gamma_2 = \pi$. With the absence of noise, the system goes from splay state to full synchrony passing through a region where SCPS is stable. Nevertheless,

3.3. Microscopic approach

as proved numerically and semi-analytically, SCPS loses stability via a subcritical Hopf bifurcation and there is a large region where the system converges to the heteroclinic cycle (HC). In Figure 3.1(a-c) we report again different order parameters corresponding to such set of bifurcations using microscopic simulations with $N = 1000$ oscillators. The red circles indicate the already commented results of the deterministic case, whereas black crosses correspond to the stochastic biharmonic model with $D = 1.2 \times 10^{-4}$. The presence of noise turns SCPS stable from the splay state to the fully synchronous regime, and there is no signature of HC. Also, the transition from SCPS to full synchrony, expected around $\gamma_1 \approx 1.159$, is smoothed out. We anticipate that this is because it is no longer a true bifurcation: as discussed in the next section, under the presence of stochastic forces, it is not possible to reach a fully synchronous regime since the microscopic and macroscopic frequencies always differ from one another. Additionally, in Fig. 3.1(b) one can see a curious finite-size effect: in the asynchronous (splay-state) regime, $\tilde{P}_D(1)$ is not strictly zero for finite N . One can thereby determine its phase and compute the corresponding growth rate, which, in the presence of noise, coincides with the frequency Ω_D of the relaxation oscillations, whereas no such oscillations are generated when $D = 0$.

The difference between the bifurcation scenario of the deterministic case and the results with $D = 1.2 \times 10^{-4}$ indicate that, for intermediate values of D , a bifurcation where SCPS gets stabilized must occur. Therefore, we study a phase diagram of the stochastic biharmonic model in a parameter space that includes the noise strength. For simplicity, throughout this chapter we stick to the case where $\gamma_2 = \pi$. The results of detailed simulations performed for different values of γ_1 and D (and different system sizes) are reported in Fig. 3.1(d). Five different regions divide the resulting bifurcation diagram. The synchronous regime and the splay state, which are bounding the diagram, will be analyzed in Section 3.4. In between, region A corresponds to stable SCPS, which becomes the only stable regime between synchrony and splay state if D is large enough. Region B indicates stable switching dynamics (SD), a stochastic version of HC whose behavior is explained in the next section. Finally, the two latter regimes coexist and are stable within region C. The rest of this chapter is dedicated to analyze in detail the different regimes and bifurcations

illustrated in this diagram.

3.3.1 Switching dynamics and SCPS

The presence of noise does not only modify the stability of the different regimes, it also changes their properties with respect to the deterministic case. Figure 3.2(a,b) shows a snapshot of the probability density $P_D(\theta)$ for two different points in parameter space, both falling in the region where SCPS is unstable for $D = 0$ (see the green circles in Fig. 3.1(d)). Panel (a) corresponds to the point inside region A: here, the probability density shifts rigidly with a constant velocity, as expected for SCPS. At variance with the deterministic case, the microscopic quasiperiodicity is lost, due to the presence of noise, but it still holds true that the microscopic average frequency differs from the macroscopic one. Panel (b) corresponds to the green point inside region B: here, the distribution is bimodal and the two peaks breath – a reminiscence of the different stability of the two clusters. The properties of this SD regime in this very same model had been studied in detail by Hanso, Mato, and Meunier [8].

A more accurate characterization of SCPS and SD is obtained by looking at the time evolution of the order parameter $|\tilde{P}_D(1.t)|$. Figure 3.2(c) illustrates its trace for three points, all corresponding to the same γ_1 -value and different noise amplitudes (see the purple squares in Fig. 3.1(d)). The black curve corresponds to zero noise (again, a very small quenched randomness has been added to avoid the spurious collapse onto a two-cluster state). Strong, fluctuations are observed, associated to the alternation of contraction and expansion of the cluster widths. Upon increasing the noise amplitude, periodic oscillations are still observed, the amplitude of which decreases while their period shrinks (see the red curve in Fig. 3.2(c)). A logarithmic reduction of the period was already noticed in [8]: it is due to the fact that the noise prevents a cluster to become too thin. The additional fluctuations are finite-size effects which decrease upon increasing the system size. For a still small but larger noise ($D = 1.2 \times 10^{-4}$), the oscillations practically disappear: the fluctuations exhibited by the blue continuous curve are just manifestations of finite-size

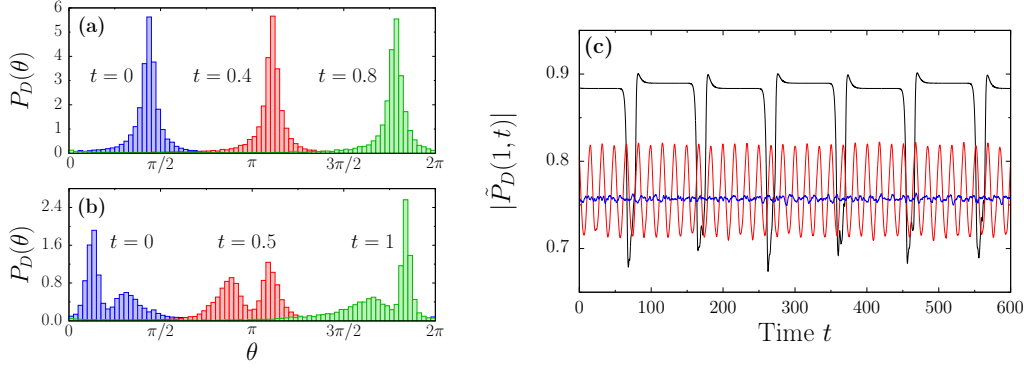


Figure 3.2: (a,b) Histograms for the density of oscillators at different times. System parameters are $\gamma_1 = 1.25$ and $D = 2 \cdot 10^{-5}$ in panel (a), and $\gamma_1 = 1.35$ and $D = 5 \cdot 10^{-5}$ in panel (b). The results have been obtained from direct simulations using 10^4 oscillators after discarding a transient of 10^5 time units. (c) Time series of the evolution of the Kuramoto order parameter for different values of noise and fixed $\gamma_1 = 1.34$ and $\gamma_2 = \pi$. Black line is the evolution of the deterministic system, red line corresponds to $D = 6 \cdot 10^{-5}$ and blue continuous line to $D = 1.2 \cdot 10^{-4}$. Results correspond to simulations with $N = 10^4$ oscillators after discarding a transient of length 9×10^4 .

effects. In fact, for this noise amplitude any reminiscence of SD is lost, as confirmed by the series of crosses plotted in Fig. 3.1(a-c).

3.3.2 Noise-induced bifurcations

Altogether, the direct simulations suggest that the transition from SCPS to SD corresponds to a Hopf bifurcation, beyond which a constant order parameter starts exhibiting periodic oscillations, which are reminiscent of the presence of the unstable two-cluster state. One can interpret SD as a sort of more structured SCPS, since also in this case there is a difference between the microscopic and macroscopic frequency. In order to validate this interpretation, we have determined the time averaged Kuramoto order parameter $\langle |\tilde{P}_D(1)| \rangle$ (we dropped the time dependence for simplicity) and the mean-field frequency Ω_D for $\gamma_1 = 1.34$ and an adiabatic increase of D from region B to A along the left dashed line in Fig. 3.1(d). Figure 3.3(a) shows that $\langle |\tilde{P}_D(1)| \rangle$ progressively decreases upon increasing the noise: this is because noise tends to glue together the two clusters. A signature of a

Chapter 3. Noise-induced stabilization of collective dynamics

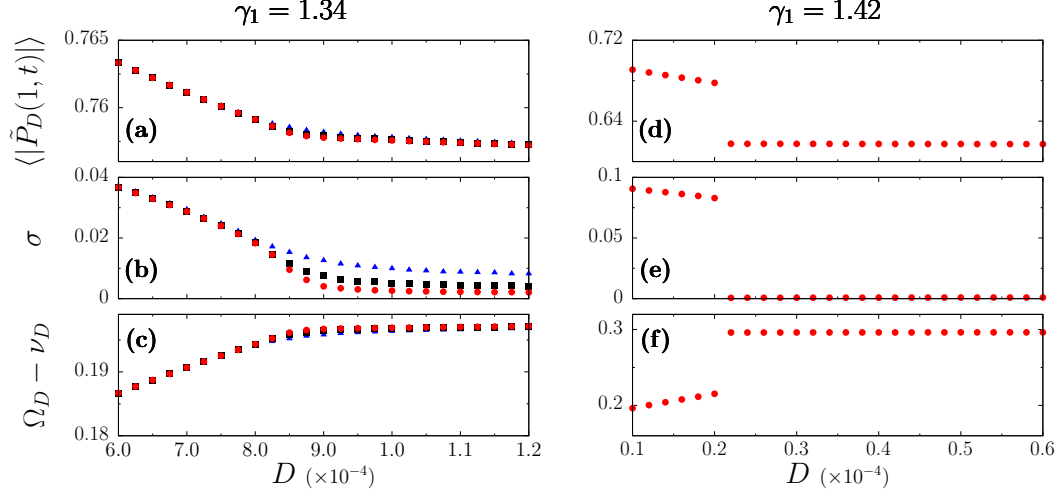


Figure 3.3: Different order parameters extracted from direct simulations of the biharmonic model versus the level of noise for $\gamma_1 = 1.34$ in panels (a),(b), and (c); and $\gamma_1 = 1.42$ in panels (d),(e), and (f). Panels (a,d) show the time-averaged Kuramoto order parameter $\langle \tilde{P}_D(1) \rangle$; panels (b,e) show its standard deviation σ ; panels (c,f) show the difference between the macroscopic frequency Ω_D and the averaged microscopic frequencies ν_D . Blue triangles, black squares, and red circles in panels (a-b) correspond to $N = 1000$, 4000 and 16000 oscillators respectively. Panels (d-f) correspond to simulations with $N = 10^4$ oscillators.

true transition can be seen in panel (b) where the (temporal) standard deviation of $|\tilde{P}_D(1)|$,

$$\sigma = \sqrt{\left\langle \left(|\tilde{P}_D(1)| - \langle |\tilde{P}_D(1)| \rangle \right)^2 \right\rangle},$$

is plotted for different system sizes. Upon increasing N , σ clearly approaches zero above a critical noise strength. Finally, panel (c) shows that $\Omega_D - \nu_D$ is different from zero both, above and below the bifurcation, confirming that the qualitative difference between SD and SCPS is just the periodic modulation of the Kuramoto order parameter.

This transition scenario from SD to SCPS remains essentially unchanged so long as $\gamma_1 \lesssim 1.4$. For larger γ_1 -values, in region C, both SCPS and SD are stable for arbitrarily small noise, so that one expects the Hopf bifurcation separating B from C to be subcritical. This observation is confirmed by the discontinuous jumps observed in Fig. 3.3(d-f), where the average order parameter, the standard deviation σ , and the frequency difference are

plotted while adiabatically increasing D starting from the SD regime (along the right vertical dashed line in Fig. 3.1(d)). In practice SD suddenly disappears through a saddle-node bifurcation, where it collides with a similar unstable regime.

Upon increasing D , these two bifurcation lines bounding region C merge together in a tricritical point, a Bautin bifurcation. The identification of this point requires some extra effort since, close to it, the amplitude of the discontinuity progressively vanishes. By comparing simulations performed by adiabatically increasing and decreasing γ_1 , we estimate the tricritical point to be located at $\gamma_1 \simeq 1.3885$, $D \simeq 7 \times 10^{-5}$.

3.4 Macroscopic description

The addition of noise on system (3.2) modifies also its treatment in the limit of infinite oscillators, $N \rightarrow \infty$. In this case the probability density of the phases, $P_D(\phi, t)$, is no longer described by the Liouville operator, but by a Fokker-Plank equation. As in the previous chapter, we analyze the equation for P_D in a frame rotating with frequency Ω_D , *i.e.*, $\theta = \phi - \Omega_D t$,

$$\frac{\partial}{\partial t} P_D(\theta, t) = \frac{\partial}{\partial \theta} \left[\left(\Omega_D - B[\theta, \tilde{P}_D(1, t), \tilde{P}_D(2, t)] \right) P_D(\theta, t) \right] + D \frac{\partial^2}{\partial \theta^2} P_D(\theta, t) , \quad (3.3)$$

where D is the diffusion term, and

$$\begin{aligned} B[\theta, \tilde{P}_D(1, t), \tilde{P}_D(2, t)] &:= \int_0^{2\pi} d\psi G(\psi - \theta) P_D(\psi, t) \\ &= -\frac{i}{2} \left(\tilde{P}_D(1, t) e^{-i(\theta - \gamma_1)} + a \tilde{P}_D(2, t) e^{-i(2\theta - \gamma_2)} - \text{c.c.} \right) \end{aligned} \quad (3.4)$$

is the velocity field.

3.4.1 Splay state

The analysis of the splay state in the stochastic version of the model is analogous to that of the deterministic model. The splay state is characterized by $P_D(\theta, t) = 1/(2\pi)$. Let us consider an infinitesimal perturbation $u(\theta, t)$ of P_D . The corresponding linearized equation is

$$\frac{\partial}{\partial t} u(\theta, t) = -\frac{1}{2\pi} \frac{\partial}{\partial \theta} B[\phi, \tilde{u}(1, t), \tilde{u}(2, t)] + D \frac{\partial^2}{\partial \theta^2} u(\theta, t) \quad (3.5)$$

where we have used that $B[\theta, \tilde{P}_D(1), \tilde{P}_D(2)] = 0$, and

$$\tilde{u}(k, t) = \int_0^{2\pi} u(\theta, t) e^{ik\theta} d\theta,$$

are the Fourier modes of u . Using the definition of B given in Eq. (3.4) one has

$$\frac{\partial}{\partial \theta} B[\theta, \tilde{u}(1, t), \tilde{u}(2, t)] = \frac{1}{2} (\tilde{u}(1, t) e^{-i(\theta-\gamma_1)} + 2a\tilde{u}(2, t) e^{-i(2\theta-\gamma_2)} + \text{c.c.}).$$

Inserting this expression in Eq. (3.5) and expanding u in Fourier space one obtains

$$\begin{aligned} \frac{\partial u}{\partial t}(\theta, t) &= \frac{1}{2\pi} \frac{\partial}{\partial t} \sum_{k=-\infty}^{+\infty} \tilde{u}(k, t) e^{-i\theta k} \\ &= \frac{1}{4\pi} (\tilde{u}(1, t) e^{-i(\theta-\gamma_1)} + 2a\tilde{u}(2, t) e^{-i(2\theta-\gamma_2)} + \text{c.c.}) - \frac{D}{2\pi} \sum_{k=-\infty}^{+\infty} k^2 \tilde{u}(k, t) e^{-i\theta k}. \end{aligned}$$

As in the deterministic case, the Fourier modes diagonalize, so one can write explicit equations for each $\tilde{u}(k, t)$,

$$\begin{aligned} \frac{\partial \tilde{u}(1, t)}{\partial t} &= \left(\frac{1}{2} e^{i\gamma_1} - D \right) \tilde{u}(1, t), \\ \frac{\partial \tilde{u}(2, t)}{\partial t} &= (a e^{i\gamma_2} - 4D) \tilde{u}(2, t), \\ \frac{\partial \tilde{u}(k, t)}{\partial t} &= -Dk^2 \tilde{u}(k, t) \quad \text{if } |k| > 2 \end{aligned}$$

3.4. Macroscopic description

complemented by the complex conjugate equations for the negative k -modes. Since the eigenvalues corresponding to the eigenfunctions with $|k| > 2$ are negative real numbers, the stability is determined only by the eigenvalues corresponding to the two first Fourier modes,

$$\lambda_1 := \frac{1}{2}e^{i\gamma_1} - D \quad \text{and} \quad \lambda_2 := ae^{i\gamma_2} - 4D$$

In the particular case $\gamma_2 = \pi$, the second eigenvalue has $\text{Re}(\lambda_2) < 0$ so that the stability of the splay state is controlled only by the first mode,

$$\text{Re}(\lambda_1) = \frac{1}{2} \cos \gamma_1 - D < 0.$$

The critical curve where $\text{Re}(\lambda_1) = 0$ corresponds to the right almost vertical line reported in Fig. 3.1(d). Notice that if $D > 0.5$ the splay state turns stable for all values of γ_1 and γ_2 , since the stochasticity is much stronger than the internal dynamics of the oscillators.

3.4.2 Nonexistence of other incoherent solutions

In the previous chapter we showed that any solution satisfying $\tilde{P}(1, t) = \tilde{P}(2, t) = 0$ is a stationary distribution of the deterministic biharmonic model. Nevertheless, as soon as $D > 0$ this is not true anymore: the condition that the first and second Fourier modes are zero ensures that the first term on the r.h.s. of Eq. (3.3) vanishes, but the diffusive term remains. Therefore, if $P(\theta, 0)$ is an initial condition with $\tilde{P}_D(1, 0) = \tilde{P}_D(2, 0) = 0$, then its time evolution is determined by.

$$\frac{\partial}{\partial t} P_D(\theta, t) = D \frac{\partial^2}{\partial \theta^2} P_D(\theta, t) .$$

This linear partial differential equation can be solved easily in Fourier space, so that

$$\tilde{P}(k, t) = \tilde{P}(k, 0)e^{-k^2 t} \quad \text{for} \quad |k| > 2 .$$

i.e., all the Fourier modes with $|k| > 2$ relax exponentially to zero due to the diffusion. In other words, noise kills all the higher harmonics of any initial incoherent distribution, so that they always end up in the splay state.

3.4.3 The synchronous state and self-consistent partial synchrony

Synchrony and noise

In the deterministic case, the fully synchronous state is characterized by a δ -like distribution and there is a well defined stability boundary for this solution. In order to understand what happens once noise is added, it is convenient to look at Eq. (3.3) as if the velocity field was given *a priori*. It corresponds to a standard Fokker-Planck equation in a washboard potential defined as

$$\frac{\partial V(\theta)}{\partial \theta} = \Omega_D - B[\theta, \tilde{P}_D(1, t), \tilde{P}_D(2, t)] ,$$

so that

$$V(\theta) = \Omega_D \theta - \left(\frac{1}{2} \tilde{P}_D(1) e^{-i(\theta - \gamma_1)} + \frac{a}{4} \tilde{P}_D(2) e^{-i(2\theta - \gamma_2)} - \text{c.c.} \right) , \quad (3.6)$$

apart from an arbitrary additional constant. Because of the tilting, as soon as $D > 0$, any stationary solution is characterized by a non-zero flux H_D , even if the potential has well defined minima. In other words, the fully synchronous state becomes formally equivalent to SCPS and no transition can be any longer found. Nevertheless, one can still identify a sort of critical line separating the regime where the potential has local minima and the flux is thereby driven by the noise, from the one where no minima exist and the flux is the result of a deterministic current. Figure 3.4 shows how the potential $V(\theta)$ changes shape upon varying γ_1 for a fixed noise strength (the *unknown* parameters Ω_D , $\tilde{P}_D(1, t)$ and $\tilde{P}_D(2, t)$ have been determined with the help of direct simulations). Using this technique we have reconstructed the “transition” line reported in Fig. 3.1(d) (see the left quasi-vertical line).

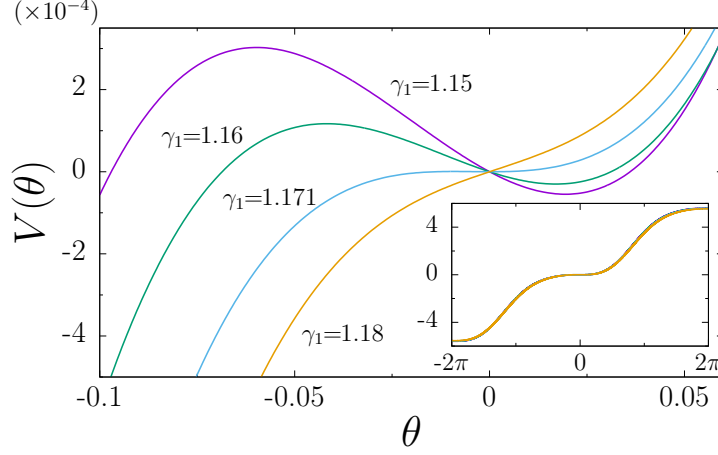


Figure 3.4: Washboard potentials corresponding to Eq. (3.6) computed for a noise level of $D = 6 \cdot 10^{-5}$. From top left to bottom left purple, green, blue and amber correspond to $\gamma_1 = 1.15, 1.16, 1.171$ and 1.18 respectively. Inset shows the same results in a enlarged parameter range. The parameters used in each case have been computed numerically from direct simulations with $N = 10^4$ oscillators.

A perturbative approach towards SCPS

As we did in the previous chapter, assuming that P_D is stationary in a suitable frame rotating with frequency Ω_D , the temporal derivative of the l.h.s. in the Fokker-Plank equation (3.3) vanishes. Integrating the resulting expression one gets

$$-H_D = \left(\Omega_D - B[\theta, \tilde{P}_D(1, t), \tilde{P}_D(2, t)] \right) P_D(\theta, t) + D \frac{\partial}{\partial \theta} P_D(\theta, t), \quad (3.7)$$

where H_D is the probability flux,

$$H_D = \frac{\nu_D - \Omega_D}{2\pi}.$$

If $D > 0$ the previous equation is an ODE. In principle, it is possible to solve such equation analitically by introducing a Jacobi-Anger expansion. Nevertheless, such expression would involve series with terms depending on Bessel functions, making both the analytic and numerical treatment highly complex and, thus, inappropriate to effectively study P_D

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and its stability properties. It is more convenient to develop a perturbative formalism to investigate the problem in a semi-analytic way in the limit of small noise, *i.e.*, $D \ll 1$.

In the previous chapter we have determined the shape and stability of $P_D(\theta)$ in the deterministic case, $D = 0$. We recall that the resulting shape of P_0 is given by

$$P_0(\theta) = \frac{-H_0}{\Omega_0 - B[\theta, \tilde{P}_0(1), \tilde{P}_0(2)]} . \quad (3.8)$$

This solution is valid if the denominator has no zeros, *i.e.*, if there are no minima in the corresponding potential. In the previous chapter we showed that this expression can be solved self-consistently by determining the parameters Ω_0 , $\tilde{P}_0(1)$, and $\tilde{P}_0(2)$ numerically. On the other hand, the probability flux H_0 is automatically determined by imposing normalization condition of P_0 .

At first order, the probability density can be written as $P_D(\phi) = P_0(\phi) + Dp(\phi)$. Moreover, since one expects variations of the macroscopic frequency as well as of the flux, we assume $\Omega_D = \Omega_0 + D\omega$ and $H_D = H_0 + D\eta$. Upon replacing these assumptions into Eq. (3.7) and retaining first order corrections in D , it is found that

$$\eta = \frac{H_0}{P_0(\theta)} p(\theta) - P'_0(\theta) - \left\{ \omega - B[\theta, \tilde{p}(1), \tilde{p}(2)] \right\} P_0(\theta)$$

where $\tilde{p}(k)$ stands for the k -th Fourier mode of p , while the prime denotes a derivative with respect to θ . This equation can be easily solved for $p(\theta)$, obtaining

$$p(\theta) = \frac{P_0(\theta)}{H_0} \left(\eta + P'_0(\theta) + \left\{ \omega - B[\theta, \tilde{p}(1), \tilde{p}(2)] \right\} P_0(\theta) \right) . \quad (3.9)$$

In order to complete the identification of the solution, it is necessary to determine $\tilde{p}(1)$, $\tilde{p}(2)$ and the two scalars ω , η . Since $\tilde{P}(1)$ in equation (3.8) is real, so has to be $\tilde{p}(1)$, while η can be obtained by imposing the “normalization” condition $\int p(\psi) d\psi = 0$. Therefore one is facing a problem of the same complexity as Eq. (3.8), and it can be solved using similar

3.4. Macroscopic description

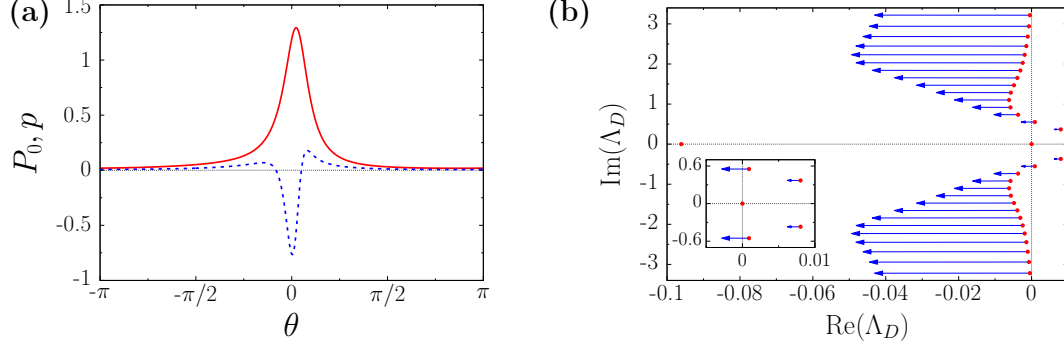


Figure 3.5: (a) Shape of $P_0(\theta)$ (red line) and $p(\theta)$ (blue dashed line) for $\gamma_1 = 1.33$. The noise-induced perturbation p has been arbitrarily rescaled. These functions correspond to the solutions of Eq. (3.8) and Eq. (3.9) respectively. (b) Red circles show real and imaginary part of the eigenvalues controlling stability of $\tilde{P}$ for $\gamma_1 = 1.33$ and $D = 0$. Blue arrows show the directions where the eigenvalues are pushed when $D = 1.7 \cdot 10^{-5}$. The inset is a zoom for $\text{Re}(\Lambda_D) > 0$.

procedures. An example of P_0 and of the corresponding variation p obtained by solving Eqs. (3.8,3.9), is shown in Fig. 3.5(a). From the shape of p , one can see that the effect of noise is to deplete the left shoulder of the density and to raise a bit its tails.

Stability analysis of SCPS

An accurate estimate of P_D allows for a reliable stability analysis. In fact, only the stationary solution is marginally stable against a rigid translation, which should be reflected in our results.

Upon linearizing Eq. (3.3) around $P_D(\theta)$, one finds that an infinitesimal perturbation $u(\theta, t)$ satisfies the equation

$$\begin{aligned} \frac{\partial}{\partial t} u(\theta, t) = & \frac{\partial}{\partial \theta} \left(\left\{ \Omega_D - B[\theta, \tilde{P}_D(1), \tilde{P}_D(2)] \right\} u(\theta, t) \right. \\ & \left. - P_D(\theta) B[\theta, \tilde{u}(1, t), \tilde{u}(2, t)] \right) + D \frac{\partial^2}{\partial \theta^2} u(\theta, t) . \end{aligned}$$

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As in the deterministic case, the spectrum of this linear operator is purely point-like and can be determined by approximating the infinite-dimensional operator with finite matrices of increasing size. The most effective method consists in expanding $u(\theta, t)$ into Fourier modes

$$u(\theta, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{u}(k, t) e^{-ik\theta}.$$

By making use of Eq. (3.4), one can rewrite the evolution equation as

$$\dot{\tilde{\mathbf{u}}} = [\mathcal{S}(\tilde{\mathbf{P}}_{\mathbf{D}}, \Omega) - D\mathcal{C}] \tilde{\mathbf{u}} \quad (3.10)$$

where $\tilde{\mathbf{u}} = \{\tilde{u}(k, t)\}_{k=-\infty}^{\infty}$, $\tilde{\mathbf{P}}_{\mathbf{D}} = \{\tilde{P}_D(k)\}_{k=-\infty}^{\infty}$, while $\mathcal{S}$ and $\mathcal{C}$ are two matrices defined as follows

$$\begin{aligned} \mathcal{S}_{mn}(\tilde{\mathbf{P}}_{\mathbf{D}}, \Omega_D) := m & \left\{ -i\Omega_D \delta_{m,n} \right. \\ & + \frac{1}{2} e^{i\gamma_1} [\tilde{P}_D(1) \delta_{m-1,n} + \tilde{P}_D(m-1) \delta_{1,n}] \\ & - \frac{1}{2} e^{-i\gamma_1} [\tilde{P}_D(-1) \delta_{m+1,n} + \tilde{P}_D(m+1) \delta_{-1,n}] \\ & + \frac{1}{2} a e^{i\gamma_2} [\tilde{P}_D(2) \delta_{m-2,n} + \tilde{P}_D(m-2) \delta_{2,n}] \\ & \left. - \frac{1}{2} a e^{-i\gamma_2} [\tilde{P}_D(-2) \delta_{m+2,n} + \tilde{P}_D(m+2) \delta_{-2,n}] \right\}, \end{aligned}$$

$$\mathcal{C}_{mn} = m^2 \delta_{m,n},$$

By recalling that $\tilde{P}_D(m) = \tilde{P}_0(m) + D\tilde{p}(m)$, we can insert this perturbative expression into Eq. (3.10), obtaining

$$\dot{\tilde{\mathbf{u}}} = \mathcal{S}(\tilde{\mathbf{P}}_0, \Omega_0) \tilde{\mathbf{u}} + D(\mathcal{S}(\tilde{\mathbf{p}}, \omega) - \mathcal{C}) \tilde{\mathbf{u}}.$$

The stability of SCPS is determined by the eigenvalues $\{\Lambda_D^{(k)}\}_k$ of the linear operator defined by this equation. Observe that $\dot{\tilde{u}}(0, t) = 0$, while $\tilde{u}(0, t)$ is decoupled from all other

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Fourier modes. This is a consequence of norm conservation, which implies that one zero eigenvalue is always present. Another obvious property of the linear operator is that exchanging positive with negative components is equivalent to taking the complex conjugate: this is due to the probability density being a real quantity.

In the limit $D \rightarrow 0$ the evolution operator reduces to $\mathcal{S}(\tilde{\mathbf{P}}_0, \Omega_0)$. Its eigenvalues $\Lambda_0^{(k)}$ are responsible for the instability of SCPS in the zero-noise limit, as studied in the previous chapter. The perturbative corrections to the evolution operator defined in Eq. (3.10) are composed of two contributions: the first one is due to the change of shape of the probability density induced by noise; the second one is the direct consequence of the diffusion term, which strongly damps short-wavelength Fourier modes (see the negative diagonal terms of $\mathcal{C}$ proportional to m^2). The eigenvalues $\{\Lambda_D^{(k)}\}_k$ can be determined perturbatively by following standard procedures. Upon expanding the eigenvalues up to first order in D , $\Lambda_D^{(k)} = \Lambda_0^{(k)} + D\lambda^{(k)}$, one can indeed write [10]

$$\lambda^{(k)} = \frac{\mathbf{W}_0^{(k)T} [\mathcal{S}(\tilde{\mathbf{p}}, \omega) - \mathcal{C}] \mathbf{V}_0^{(k)}}{\mathbf{W}_0^{(k)T} \mathbf{V}_0^{(k)}}$$

where $\mathbf{W}_0$ and $\mathbf{V}_0$ are the left-, resp. right-hand eigenvectors associated to $\Lambda_0^{(k)}$. In principle, the matrix indices range from $-\infty$ to $+\infty$. One can however obtain sufficiently accurate estimates of the largest eigenvalue by restricting the analysis to a finite range $[-M, M]$, provided that M is large enough. We have verified that $M = 100$ suffices to reconstruct the relevant part of the stability spectrum.

Using the numerical solutions of Eqs. (3.8,3.9) for the shape of the stationary probability density of the Fokker-Planck equation up to first order in D we are able to determine the eigenvalues of controlling the stability of SCPS in presence of noise. The results for $\gamma_1 = 1.33$ are illustrated in Fig. 3.5(b), where the eigenvalues are plotted for $D = 0$ (red dots) and $D = 1.7 \cdot 10^{-5}$ (the arrows denote the shift of each eigenvalue). The spectrum is composed of pairs of complex conjugate eigenvalues; two of them are characterized by a

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strictly zero imaginary part: the first one corresponds to the zero exponent, which follows from the invariance of the solution under a rigid phase shift and is present in the noisy regime as well; the second one is the most negative eigenvalue (see the leftmost red dot) - very weakly affected by noise.

The effect of noise is to basically shift all eigenvalues towards more negative values: in fact, all of the arrows reported in Fig. 3.5(b) are practically horizontal and point to the left. The stabilizing effect depends strongly on k (because of the diffusive term in Eq. (3.3)); it is approximately proportional to k^2 . In particular, it is very weak for the two pairs of unstable eigenvalues (see also the inset). Nevertheless, even for a maximally unstable γ_1 (the value chosen in Fig. 3.5(b)), a tiny amount of noise is sufficient to stabilize one of the two pairs of unstable directions (a noise amplitude about 7 times larger would fully stabilize SCPS). Upon tracking the eigenvalues with positive real part for increasing D one can identify where the stabilization of SCPS takes place. The transition points are plotted as blue triangles in Figure 3.1(d) and show a good agreement with the direct numerical simulations.

3.5 Conclusions

In the previous section we have performed a detailed stability analysis of SCPS in the presence of noise. This collective regime corresponds to a stationary state in a suitably moving frame, where its evolution is described by a nonlinear Fokker-Planck equation. The nonlinearities play a double role: on the one hand they contribute to a self-consistent determination of the underlying potential; on the other hand they contribute to the stability of the state itself. Indeed, in the deterministic case, SCPS loses stability through a subcritical Hopf bifurcation. In this chapter we have proven that a very small noise is sufficient to stabilize such collective dynamics.

It is therefore natural to ask whether this scenario is peculiar of the biharmonic setup. We check this point by using again, an ensemble of mean-field coupled Rayleigh oscilla-

tors. In the previous chapter we showed that SCPS emerges in a pure deterministic setup, although it loses stability passed a critical value. Here we show that noise is again able to stabilize SCPS. The Rayleigh oscillator model reads as

$$\ddot{x}_j - \zeta(1 - \dot{x}_j^2) + x_j = \varepsilon \operatorname{Re}[e^{i\gamma}(X + iY)] + \eta_j(t)$$

where $X = N^{-1} \sum_m x_m$ and $Y = N^{-1} \sum_m \dot{x}_m$ are the mean field contributions to the coupling, and $\varepsilon = 0.05$ is the coupling strength. We assume again white noise $\eta_j(t)$ with $\langle \eta_j(t) \rangle = 0$ and $\langle \eta_j(t) \eta_m(t') \rangle = 2D \delta_{jm} \delta(t - t')$. The parameter ζ determines the stability of the limit-cycle, $\zeta = 5$ for there is a strong attraction. Therefore γ is the main control parameter that is going to be tuned. We use again the order parameter

$$\rho = \operatorname{rms}(X) / \operatorname{rms}(x)$$

where

$$\operatorname{rms}(x) = \sqrt{\langle x(t)^2 \rangle}$$

is the root mean square of the time evolution. Notice that $\rho = 1$ when there is fully synchrony and $\rho = 0$ when the oscillators are distributed uniformly.

In the deterministic case (see Chapter 2), in the range $\gamma \in [-0.7, 0.2]$ a wide number of dynamical regimes are observed (see red circles in Figure 3.6). In particular, SCPS is observed for $-0.18 \lesssim \gamma \lesssim 0.05$. Above $\gamma \simeq 0.05$ the system converges to a homogeneous nine-cluster state. Similarly to the biharmonic model, SCPS loses stability towards a two-cluster state for $\gamma \simeq -0.18$. This cluster state finally converges to full synchrony at $\gamma \simeq -0.57$. On the one hand, a small noise ($D = 10^{-6}$) does not substantially affect the regions where full synchrony and the nine-cluster states are stable. On the other hand, it is once again able to stabilize SCPS in the entire interval up to full synchrony (see the black crosses in Fig. 3.6).

The study of two models of phase oscillators has shown that a small amount of micro-

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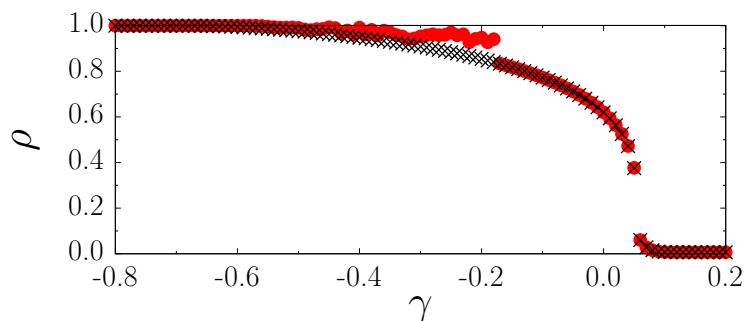


Figure 3.6: Order parameter ρ for different values of γ_1 for no noise (red circles) and $D = 10^{-6}$ (black crosses). The results have been obtained by direct simulations of the Rayleigh oscillators with 1000 units.

scopic noise stabilizes self-consistent partial synchrony. It is natural to ask whether this effect extends to other types of collective dynamics. In globally coupled identical maps, collective chaos can be observed [11]. In such a setup, it was found that an additive noise of the type considered in this chapter can reduce the dimensionality of the collective dynamics [11, 12]. Considering that a chaotic evolution can be seen as a sort of wandering process among different unstable periodic orbits, it is tempting to interpret this reduction of dimensionality as a progressive stabilization of the dynamics along various directions. It will be instructive to further investigate this interpretation.

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Chapter 4

From phase to amplitude oscillators

4.1 Introduction

Phase oscillatory systems, such as the biharmonic model, represent a simple paradigm to study the emergence of nontrivial dynamical regimes over multiple scales. Nevertheless, it is not clear how and to what extent the phase reduction approach is sufficiently general for the emergence of more complex macroscopic behavior. As an important milestone, the question whether collective chaos can emerge from globally coupled phase oscillators still stands. To answer it one needs to understand the way macroscopically active degrees of freedom spontaneously emerge in ensembles of identical units beyond the phase approximation.

It is known since many years that collective chaos may spontaneously emerge in different types of mean-field models. One can classify the distinct setups according to the dynamical complexity of the single oscillators, since the overall behavior should be ultimately traced back to the rules determining the evolution of the single elements. Intrinsically chaotic elements such as logistic maps sit at the top of the hierarchy. In this case, collective chaos arises as a result of subtle correlations among the single dynamical units [1]. Collective chaos can emerge also in periodic oscillators such as the Stuart-Landau model [2, 3, 4], since the macroscopic mean field can trigger and maintain a mi-

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croscopic chaos, thereby giving rise to a regime similar to the previous one.

By further descending the ladder of the single-unit complexity, no evidence of collective chaos has been found in mean-field models of identical phase-oscillators, but no rigorous argument excluding that this can happen is available. In fact, in all mean-field systems of identical oscillators, the probability distribution satisfies a self-consistent functional equation which, being nonlinear and infinite-dimensional can in principle give rise to a chaotic dynamics. Nevertheless, collective chaos in phase oscillators has been found only when either delayed interactions [5], multiple populations [6], or heterogeneity [7] is included in the model. Otherwise, at most macroscopic periodic oscillations arise (the so-called self-consistent partial synchrony SCPS) [8, 9, 10, 11], the simplest setup for their observation being the biharmonic Kuramoto-Daido model studied in the previous chapters. The so-called balanced regime observed in the context of neural networks is, instead, an entirely different story, since the collective dynamics is basically the result of the amplification of microscopic fluctuations [12].

Altogether, new theoretical approaches are required to improve our general understanding of the collective behavior of ensembles of dynamical units. In phase oscillators characterized by strictly sinusoidal coupling-functions (Kuramoto type) the Watanabe-Strogatz theorem [13] and the Ott-Antonsen Ansatz [14] imply that no more than three collective degrees of freedom can emerge. What can one say in more general contexts? In this chapter, we discuss an intermediate regime, where amplitude oscillators (characterized by at least two variables) retain a key property of phase oscillators, *i.e.*, they can be parametrized by a single phase-like variable. In practice, the amplitude dynamics is sufficiently stable to force the oscillators towards a smooth closed curve $\mathcal{C}$ such as in standard phase oscillators, but not stable enough to prevent fluctuations of the curve itself, which thereby acts as an additional source of complexity. We refer to this regime, which has been so far basically overlooked, as to Quasi Phase Oscillators (QPO). We are only aware of a preliminary study carried out by Kuramoto & Nakagawa, who performed microscopic simulations of an ensemble of Stuart-Landau oscillators [15]. Here we develop

a general formalism, which allows describing QPO at a macroscopic level. As a result, we are able to identify and describe a series of dynamical regimes ranging from periodic SCPS to a new type of collective chaos, passing through other novel types of SCPS with larger macroscopic complexity.

SCPS itself is already an intriguing regime. In fact, under the action of a self-consistent mean field, the single oscillators exhibit behavior without phase locking (no Arnold tongues). Said differently, the transversal Lyapunov exponent of each single oscillator (conditioned to a given “external” forcing) is consistently equal to zero, when a control parameter is varied. The hallmark of QPO is that the transverse Lyapunov exponent is not positive, even when the “external” mean-field forcing is chaotic: actually, it is even slightly negative, this implying that the phase-probability distribution is not smooth but multifractal. This is at variance with the “standard” collective chaos observed in a different parameter region of the Stuart-Landau model [2, 16, 3, 4], characterized by a positive transversal Lyapunov exponent.

4.2 The model

Following Ref. [2], we consider an ensemble of N coupled Stuart-Landau oscillators

$$\dot{z}_j = K(1 + ic_1)(\bar{z} - z_j) + z_j - (1 + ic_2)|z_j|^2 z_j \quad (4.1)$$

where $z \in \mathbb{C}$ and

$$\bar{z} = \frac{1}{N} \sum_{m=1}^N z_m$$

is the mean field. The formulation of Refs. [4, 3] is recovered by setting $t \rightarrow Kt$, $\alpha = c_2$, $\eta = c_1$ and $\mu = K^{-1}$.

As in phase models, a homogeneous system of globally coupled oscillators such as (4.1)

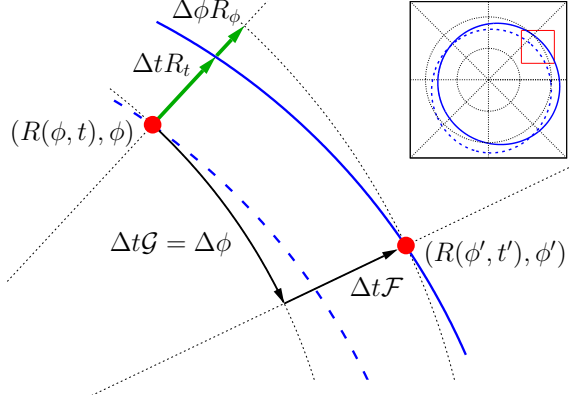


Figure 4.1: Schematic representation of the phase parametrization of $\mathcal{C}$ and its time evolution. Dashed blue curve depicts the shape of $\mathcal{C}$ at time t , whereas the continuous blue line indicates the position of $\mathcal{C}$ at time $t' = t + \Delta t$. The inset shows a zoom out version, with the red rectangle indicating the zone depicted in the main figure.

can display two types of stationary solutions: a state of *full synchrony* and *incoherent states*. In the state of full synchrony all the oscillators evolve periodically as $z_j = \bar{z} = e^{ic_2 t}$. Incoherent states correspond to a distribution of oscillators such that the mean field $\bar{z}$ vanishes. There are infinitely many ways of realizing an incoherent state, the simplest being a completely homogeneous distribution of the phases along a circle (of radius $\sqrt{1 - K}$) also known as *splay state*.

For small coupling strength $K \ll 1$, the model can be mapped onto a Kuramoto-Sakaguchi model [11], which is known to give rise only to either fully synchronous or splay states. For larger coupling strength, the model displays a rich variety of dynamics, including clustered states, chimera states, and different forms of collective chaos [2, 3, 4, 15, 17]. In particular, Nakagawa and Kuramoto reported numerical evidence of a dynamical regime where the oscillators seem to be distributed along a smooth time-dependent closed curve [15]. In this chapter, we revisit such a behavior, developing a formalism, which can, in principle, be applied to generic ensembles of mean-field driven amplitude oscillators.

A general macroscopic formulation of system (4.1) requires the knowledge of the probability density on the complex plane $Q = Q(z, t)$. Nevertheless, if the amplitude dynamics is sufficiently stable, the oscillator variables naturally converge and are eventually confined to a closed curve $\mathcal{C}$. Whenever this is the case, $\mathcal{C}$ can be uniquely parametrized in polar coordinates by expressing the radius $R(\phi, t)$ as a function of the phase ϕ . Accordingly one can assume that the probability density is contained in $\mathcal{C}$ so that

$$Q(z, t) = Q(z, t) \delta(|z| - R(\phi, t)) =: P(\phi, t).$$

The meaningfulness of this approach is verified a posteriori by the consistency of the theoretical predictions and by the agreement with direct numerical simulations.

Our first goal is to derive the evolution equations for R and P . In order to do so, it is convenient to express Eq. (4.1) in polar coordinates, $z = re^{i\phi}$,

$$\begin{aligned}\dot{r}_j &= \mathcal{F}[r_j, \phi_j, \bar{z}] \\ \dot{\phi}_j &= \mathcal{G}[r_j, \phi_j, \bar{z}],\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}[r, \phi, \bar{z}] &= (1 - K - r^2)r + K \operatorname{Re}[(1 + ic_1)\bar{z}e^{-i\phi}] \\ \mathcal{G}[r, \phi, \bar{z}] &= -c_1K - c_2r^2 + \frac{K}{r} \operatorname{Im}[(1 + ic_1)\bar{z}e^{-i\phi}].\end{aligned}$$

Let us now consider a time interval $\Delta t \ll 1$ and $t' = t + \Delta t$. At time t the oscillator with index j has polar coordinates $[r_j(t), \phi_j(t)] = [R(\phi, t), \phi]$ while at time t' has $[r_j(t'), \phi_j(t')] = [R(\phi', t'), \phi']$, as indicated in Figure 4.1. If R is smooth one has that

$$\begin{cases} R(\phi', t') = R(\phi, t) + \Delta t \mathcal{F}[R(\phi, t), \phi, \bar{z}] \\ \phi' = \phi + \Delta t \mathcal{G}[R(\phi, t), \phi, \bar{z}]. \end{cases}$$

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Then one can write

$$\frac{R(\phi, t') - R(\phi, t)}{\Delta t} = \frac{R(\phi', t') - R(\phi, t)}{\Delta t} - \frac{\phi' - \phi}{\Delta t} R_\phi$$

so that

$$\frac{\partial R}{\partial t}(\phi, t) = \mathcal{F}[R, \phi, \bar{z}] - \mathcal{G}[R, \phi, \bar{z}] R_\phi . \quad (4.2)$$

On the other hand, the evolution of P is determined by the angular flux at $r = R(\phi, t)$,

$$\frac{\partial P}{\partial t}(\phi, t) = -\frac{\partial}{\partial \phi} \left\{ P(\phi, t) \mathcal{G}[R, \phi, \bar{z}] \right\}. \quad (4.3)$$

while the mean field is finally defined as

$$\bar{z}(t) = \int_0^{2\pi} P(\phi, t) R(\phi, t) e^{i\phi} d\phi . \quad (4.4)$$

From the expression for $\mathcal{G}[R, \phi, \bar{z}]$, one can recognize the Kuramoto-Sakaguchi structure [18] of the velocity field, with however, the important difference of the additional factor $R(\phi, t)$ in the definition of the order parameter. We will see that the time dependence of $R(\phi, t)$ (it obeys a distinct differential equation) enriches the complexity of the collective dynamics.

Overall, equations (4.2, 4.3, 4.4) represent a system of two nonlinear Partial Differential Equations (PDEs) describing the macroscopic behavior of the oscillators whenever they are spread along a closed phase-parameterized smooth curve. Such a system can be solved numerically by means of a split-step Fourier or pseudospectral method [19]. The algorithm consists in expanding R and P spatially in Fourier space,

$$P(\phi, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{P}(k, t) e^{-ik\phi} \quad \text{and} \\ R(\phi, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{R}(k, t) e^{-ik\phi}$$

where

$$\begin{aligned}\tilde{P}(k, t) &= \int_0^{2\pi} d\phi P(\phi, t) e^{ik\phi} \quad \text{and} \\ \tilde{R}(k, t) &= \int_0^{2\pi} d\phi R(\phi, t) e^{ik\phi}.\end{aligned}$$

By truncating the Fourier series for a large enough wavelength M , one can then integrate in time the different Fourier modes $\tilde{P}(k, t)$ and $\tilde{R}(k, t)$ using a standard method for ordinary differential equations. Since the computation of a nonlinear term of order q in Fourier space requires $\mathcal{O}(M^q)$ operations, it is more convenient to compute the velocity fields in real space instead. In fact, by invoking fast Fourier transform (FFT) algorithms, the computational cost of the nonlinear fluxes reduces to $\mathcal{O}(n \log(n))$, where n is the number of points of the real support. We use a fourth-order Runge-Kutta integration method with time step δt , set to 10^{-3} in most of the simulations. In order to avoid aliasing errors in the successive calls of the FFT we use the 3/2-truncation rule using a real spatial grid of $n = 2048$ points and a truncation of $M = 680$ in the Fourier expansion. Nevertheless, in the presence of chaotic dynamics (see below) the numerics become more challenging and it is necessary to increase both spatial and temporal resolutions. Therefore, for $K > 0.416$ we use $M = 1024$ and $\delta t = 0.5 \cdot 10^{-3}$, and keep $n = 2048$. In this case it is necessary to introduce a smoothing technique to prevent the growth of the aliasing errors. Thus we add a small diffusive term in equation (4.3),

$$\frac{\partial P}{\partial t}(\phi, t) = -\frac{\partial}{\partial \phi} \left\{ P(\phi, t) \mathcal{G}[R, \phi, \bar{z}] \right\} + D \frac{\partial^2}{\partial \phi^2} P(\phi, t).$$

A diffusion level of $D = 10^{-8}$ is enough to stabilize the algorithm while preserving the dynamical properties of the system.

In order to double check our numerics we have fixed two parameters as in Ref. [15] ($c_1 = -2$ and $c_2 = 3$) and treated the coupling strength K as the leading control parameter. The results of the numerical simulations are reported in Figures 4.2 and 4.3. Figure 4.2(a)

Chapter 4. From phase to amplitude oscillators

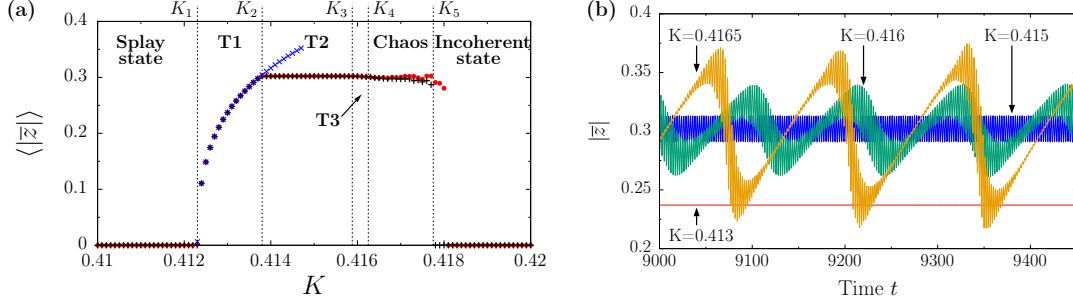


Figure 4.2: (a) Time-averaged mean-field $\langle |\bar{z}| \rangle$ for different values of the coupling strength K and fixed $c_1 = -2$ and $c_2 = 3$. Black plusses show the results from microscopic simulations of the system as given by (4.1) with $N = 4096$ oscillators computed over 10^6 time units after discarding another 10^6 of transient. Red circles correspond to the numerical integration of Eqs. (4.2) and (4.3) with a pseudospectral method for 10^5 time units, from which $5 \cdot 10^4$ are transient. Both microscopic and macroscopic simulations use initial conditions close to splay state. Blue crosses correspond to the semi-analytical fixed point given by Eqs. (4.9) and (4.10). Black vertical dashed lines indicate the bifurcation points between the different dynamical regimes. (b) Time series of $|\bar{z}|$ from the numerical integration of the macroscopic equations of the system for $K = 0.413$ (red), 0.415 (blue), 0.416 (green), and 0.4165 (amber).

shows the time-average of the mean-field $|\bar{z}|$ computed both using the microscopic formulation of the system (4.1) with $N = 4096$ (black plusses) and using the macroscopic equations solved with the pseudospectral method (red circles). The nice agreement confirms the correctness of the macroscopic equations.

The behavior of the order parameter unveils a series of bifurcations across various dynamical regimes, marked with vertical dashed black lines. Figure 4.2(b) shows the time evolution of the order parameter modulus $|\bar{z}|$ for different values of K , each belonging to a different dynamical regime. The top panels of Figure 4.3 depict snapshots of R (black dashed lines) and the corresponding probability densities P (red lines). While the shape of the curve $\mathcal{C}$ remains very smooth when the coupling is increased, the probability profiles become increasingly rugged, revealing an increasing contribution of higher Fourier modes. This is better appreciated by looking at the bottom panels of Fig. 4.3, where the corresponding Power Spectral Density (PSD) is reported in a logarithmic scale (same color code for the different curves).

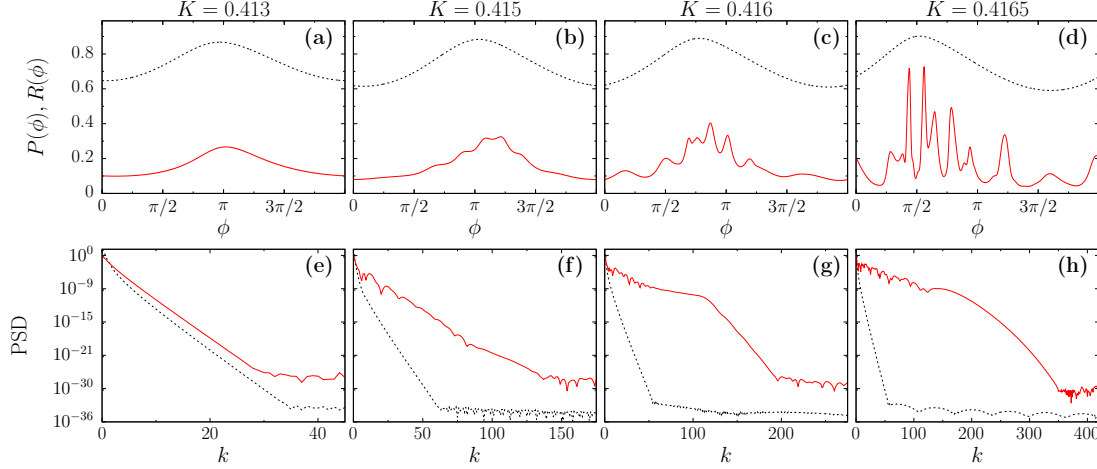


Figure 4.3: Snapshots of the shape of R (dashed black curve) and P (red continuous curve) in real space (top figures), and the corresponding Power Spectral Density in logarithmic scale (bottom figures). (a) and (e) correspond to $K = 0.413$, (b) and (f) to $K = 0.415$, (c) and (g) to $K = 0.416$, and (d) and (h) to $K = 0.4165$. Results obtained upon integrating the macroscopic equations using a pseudospectral method.

For $K < K_1 \approx 0.4123$, the oscillators are uniformly distributed on a perfect circle and the mean-field $\bar{z}$ vanishes, *i.e.*, the system is in a *splay state*. For $K = K_1$ there is a first transition to a regime where the mean-field dynamics is a pure periodic rotation on a one-dimensional torus (T1), accompanied by a quasiperiodic dynamics of the single oscillators. This regime is therefore an new example of periodic SCPS as studied in Chapter 2. Thus, in this case, the mean-field modulus is constant (see red line in Fig. 4.2(b)). In fact, both P and R are spatially nonuniform steady functions if observed in a suitably rotating frame (see Fig. 4.3(a), where one can also notice that the shape is dominated by a few long wavelength modes - panel (e)). At $K = K_2 \simeq 0.4138$ a macroscopic Hopf bifurcation occurs, which introduces a second frequency. As a result, the collective dynamics is quasiperiodic (T2, but periodic, if observed in a suitably rotating frame) (see the blue line in Fig. 4.2(b)). Now the shape of P and R fluctuate in time and more Fourier modes contribute (see Figures 4.3(b) and (f)). Thus, this is a more complex case of SCPS.

At $K = K_3 \simeq 0.41588$, yet another frequency adds to the macroscopic dynamics,

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although indistinguishable from the average value of the order parameter. Thus, as better argued in the following, the dynamics of the order parameter becomes three-dimensional (T3) (see the green line in Fig. 4.2(b)). The shape of P becomes more uneven, with several bumps that evolve on time, and the spatial spectra involve higher Fourier modes (see Fig. 4.3(c) and (g)).

The T3 regime is stable for a small range of the parameter value; beyond $K = K_4 \simeq 0.41605$ the system becomes chaotic. Although R keeps being quite smooth, the shape of P occasionally develops several peaks that become sharper upon increasing the coupling strength (see Fig. 4.3(d) and (h)). Such peaks generate spurious ringing artifacts that eventually lead to numerical instabilities. Nevertheless, as explained before, increasing the numerical resolution and introducing a negligible diffusion, one can accurately integrate the macroscopic equations for long times.

For yet larger K values, the order parameter vanishes after a long chaotic transient. However, the asymptotic regime is not a splay state: only the first Fourier mode vanishes, while the others have non zero amplitude. In other words, we are in front of non-trivial incoherent states. The disagreement between microscopic and macroscopic simulations on the bifurcation point K_5 in Figure 4.2(a) is due to the fact that macroscopic simulations cannot be run long enough for the chaotic transient to finish. Finally, if the coupling is increased even further, one ends up in the highly irregular chaotic regime studied in [2, 16], where the macroscopic equations (4.2) and (4.3) are no longer valid, since the oscillators are not distributed along a smooth closed curve.

4.3 Stability of Splay State

The macroscopic equations can be used to perform the stability analysis of the splay state by introducing the fields $u(\phi, t)$ and $v(\phi, t)$ which denote infinitesimal perturbations of the probability $P_0 = 1/(2\pi)$ and of the radius $R_0 = \sqrt{1-K}$, respectively. Upon linearizing

4.3. Stability of Splay State

Eqs. (4.2) and (4.3), it is found that

$$v_t = -2(1 - K)v + Av_\phi + K \operatorname{Re}[(1 + ic_1)\bar{w}e^{-i\phi}] ,$$

and

$$u_t = Au_\phi + \frac{c_2\sqrt{1-K}}{\pi}v_\phi + \frac{K \operatorname{Re}[(1 + ic_1)\bar{w}e^{-i\phi}]}{2\pi\sqrt{1-K}} ,$$

where

$$A = c_1K + c_2(1 - K)$$

and

$$\bar{w} := \sqrt{1-K} \int_0^{2\pi} d\phi u(\phi, t)e^{i\phi} + \frac{1}{2\pi} \int_0^{2\pi} d\phi v(\phi, t)e^{i\phi}$$

is the (linear) variation of the mean-field. It is composed of two terms: the former one is the standard Kuramoto-type order parameter, while the latter accounts for fluctuations of the curve $\mathcal{C}$.

The linearized equations can be easily solved in Fourier space, *i.e.*, by introducing

$$\begin{aligned} \tilde{v}(k, t) &= \int_0^{2\pi} d\phi v(\phi, t)e^{ik\phi} \quad \text{and} \\ \tilde{u}(k, t) &= \int_0^{2\pi} d\phi u(\phi, t)e^{ik\phi} , \end{aligned}$$

(and the corresponding inverse transforms), since they become block diagonal.

For $k \neq \pm 1$ (including $k = 0$),

$$\begin{aligned} [\tilde{v}(k)]_t &= [-2(1 - K) - ikA] \tilde{v}(k) \quad \text{and} \\ [\tilde{u}(k)]_t &= -ik \frac{c_2\sqrt{1-K}}{\pi} \tilde{v}(k) - ikA \tilde{u}(k) \end{aligned}$$

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so that, the evolution of $\tilde{v}(k)$ is closed onto itself and acts as an external forcing for the dynamics of the probability density, yielding the eigenvalues

$$\begin{aligned}\lambda_k^{(v)} &= -2(1-K) - \mathrm{i}k(Kc_1 + c_2(1-K)) \quad \text{and} \\ \lambda_k^{(u)} &= -\mathrm{i}k(Kc_1 + c_2(1-K)) \quad .\end{aligned}$$

Altogether, the eigenvalues are arranged into two branches: the former corresponds to stable directions associated to the $\mathcal{C}$ -dynamics; the latter corresponds to marginally stable directions associated to the density dynamics (this includes the zero mode, whose marginal stability is nothing but a manifestation of probability conservation). Overall the stability of the angle distribution is reminiscent of the marginal stability of the Kuramoto model.

The only exception is the first Fourier mode of $v(\phi, t)$, that is coupled back to the shape of the curve. For $k = 1$ one obtains

$$\begin{aligned}[\tilde{v}(1)]_t &= M_{vv}\tilde{v}(1) + M_{vu}\tilde{u}(1) \\ [\tilde{u}(1)]_t &= M_{uv}\tilde{v}(1) + M_{uu}\tilde{u}(1)\end{aligned}\tag{4.5}$$

where

$$\begin{aligned}M_{vv} &= -2 + \frac{5}{2}K - \mathrm{i}\left(A - \frac{Kc_1}{2}\right) \\ M_{vu} &= -\pi K\sqrt{1-K}(1 + \mathrm{i}c_1) \\ M_{uv} &= \frac{K}{4\pi\sqrt{1-K}} - \mathrm{i}\left(\frac{c_2\sqrt{1-K}}{\pi} - \frac{Kc_1}{4\pi\sqrt{1-K}}\right) \\ M_{uu} &= \frac{K}{2} - \mathrm{i}\left(A - \frac{Kc_1}{2}\right) \quad .\end{aligned}$$

Finally, the equations for $k = -1$ are obtained by simply taking the complex conjugate of the above expressions.

Accordingly, instabilities can and actually do arise within the four-dimensional sub-

4.4. Periodic self-consistent partial synchrony

space spanned by the real and imaginary parts of the first modes $\tilde{v}(1)$ and $\tilde{u}(1)$. In practice one needs to determine the stability of the two dimensional complex system (4.5). Using the Routh–Hurwitz criterion one finds two stability conditions

$$3K < 2, \quad \text{and} \\ (2c_1^2 + 8c_1c_2 - c_2^2 + 9)K^2 - (c_1^2 + 12c_1c_2 - c_2^2 + 12)K + 4 + 4c_1c_2 < 0$$

In the parameter region considered in this chapter, the second inequality reveals a loss of stability for $K > K_1 = (65 - \sqrt{1025})/80 = 0.412304\dots$, which corresponds to a pair of complex conjugate eigenvalues crossing the imaginary axis, *i.e.*, a Hopf bifurcation. This bifurcation gives rise to periodic SCPS, analyzed in the following section.

4.4 Periodic self-consistent partial synchrony

Our general formalism, based on Eqs. (4.2,4.3), allows performing an analytical study of SCPS in the T1 form using a similar approach than in Chapter 2. As in that case, this simple form of SCPS is characterized by a non-uniform probability density, rotating with a collective frequency ω , which differs from the average frequency of the single oscillators. In models where the interactions depend only of phase differences like the present one, ω is constant as well as the amplitude $|\bar{z}|$ of the order parameter. In other words, periodic SCPS corresponds to a fixed point of Eqs. (4.2,4.3) in the moving frame $\theta = \phi - \omega t$. Since the origin of the phases can be imposed freely, we select a frame where $\bar{z}$ is real and positive, *i.e.*, its phase vanishes (so that from now on, one can avoid the use of the absolute value). The equations resulting from this change of variables are

$$R_t = \mathcal{F}[R, \theta, \bar{z}] + \{\omega - \mathcal{G}[R, \theta, \bar{z}]\} R_\theta. \quad (4.6)$$

and

$$P_t = \frac{\partial}{\partial \theta} \left(P(\theta, t) \{\omega - \mathcal{G}[R, \theta, \bar{z}]\} \right). \quad (4.7)$$

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accompanied by the self-consistent condition

$$\int_0^{2\pi} d\theta P(\theta, t) R(\theta, t) e^{i\theta} =: \bar{z} . \quad (4.8)$$

By imposing the time derivative of R equal to zero in equation (4.6), we find that the stationary shape of the limit cycle, $R_0(\theta)$, can be obtained by integrating the following ODE,

$$[R_0(\theta)]_\theta = -\frac{\mathcal{F}[R_0, \theta, \bar{z}]}{\omega - \mathcal{G}[R_0, \theta, \bar{z}]} . \quad (4.9)$$

Once a (numerical) expression for R_0 has been obtained, it can be replaced in the equation for P . The stationary solution P_0 can be thereby obtained by setting the argument of the θ derivative equal to a new unknown constant: minus the probability flux η . The solution reads

$$P_0(\theta) = -\frac{\eta}{\omega - \mathcal{G}[R_0, \theta, \bar{z}]} , \quad (4.10)$$

where η can be obtained from the normalization condition of the probability density $\int_0^{2\pi} P_0 = 1$. A complete expression for $R_0(\theta)$, and $P_0(\theta)$ is finally obtained by determining the two remaining unknowns, $\bar{z}$ and ω , from the complex self-consistent condition (4.8). The resulting shapes are reported in Figs. 4.4(a,b) for two different coupling strengths. As expected, both R_0 and P_0 become increasingly nonuniform upon increasing the coupling strength. Blue crosses in Figure 4.2(a) indicate the values of $|\bar{z}|$ obtained from this approach.

4.4.1 First order approximation of SCPS

In the rotating frame, where the curve $R(\theta)$ does not depend on time, one can view the term in curly brackets in Eq. (4.7) as the force field acting on an oscillator of phase θ , so that $\mathcal{G}[R(\theta), \theta]$ plays the role of the standard coupling term in ensembles of phase oscillators. The Kuramoto-Sakaguchi model is the simplest setup where the coupling is purely sinusoidal, *i.e.*, $\mathcal{G} = G \sin(\psi + \alpha - \theta)$, where G and ψ are the amplitude and the phase of

4.4. Periodic self-consistent partial synchrony

the order parameter, respectively. In such a case, it is well known that either the splay state or the fully synchronous solution are fully stable, depending whether α is larger or smaller than $\pi/2$. Only for $\alpha = \pi/2$ intermediate macroscopically periodic solutions (SCPS-like) are possible, and, moreover, the dynamics is highly degenerate, since infinitely many solutions exist and are thereby marginally stable. However, as illustrated in Chapter 2, if a small second harmonic is added, this high degree of degeneracy is lifted and a finite parameter region appears, where robust SCPS can be observed. How do such findings compare with the scenario herein reported for QPOs?

First of all, it should be reminded that the emergence of SCPS in Stuart-Landau oscillators has been already reported in the past (see [11]), a major difference being that the theoretical and numerical studies refer to a parameter region where the single oscillators do not alter their phase character and the coupling manifests itself as a nonlinear dependence on the order parameter and, last but not least, SCPS emerges as a loss of stability of the fully synchronous state.

In the present context, the stability analysis of the splay state reveals that there is no need to include higher harmonics to correctly predict the onset of SCPS, as if they were unnecessary. Since this result conflicts with our general understanding, we have performed a perturbative expansion of the stationary solution in the vicinity of the critical point. Slightly above K_1 , the shape R_0 of the attractor and the corresponding density distribution P_0 of the phases are close to the splay state, so that one can write

$$\begin{aligned} R_0(\theta) &= \sqrt{1-K} + r(\theta) \\ P_0(\theta) &= \frac{1}{2\pi} + p(\theta) . \end{aligned} \tag{4.11}$$

Expanding the self-consistent condition (4.8) up to linear terms, one obtains

$$\bar{z} = \int_0^{2\pi} d\psi P(\psi) R(\psi) e^{i\theta}$$

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$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i\psi} d\psi r(\psi) + \sqrt{1-K} \int_0^{2\pi} d\psi p(\psi) e^{i\psi} \\
&= \frac{1}{2\pi} \tilde{r}(1) + \sqrt{1-K} \tilde{p}(1) .
\end{aligned} \tag{4.12}$$

By then expanding equation (4.9) up to first order, one obtains

$$[r(\theta)]_\theta = \frac{2(1-K)r - K\bar{z}\sqrt{1+c_1^2}\cos(\nu-\theta)}{\Delta} ,$$

where $\Delta := \omega + c_1 K + c_2(1-K)$, and $\nu := \arctan c_1$. By introducing the notations

$$a_\xi e^{i\xi} := \frac{2(1-K)}{\Delta} - i \quad \text{and} \quad A := \frac{K\sqrt{1+c_1^2}}{a_\xi \Delta} ,$$

one can write the solution of the ODE as

$$r(\theta) = \bar{z} A \cos(\nu + \xi - \theta) . \tag{4.13}$$

Accordingly, $\tilde{r}(1) = \pi \bar{z} A e^{i(\nu+\xi)}$. By expanding Eq. (4.10) in the same way, at zero order one obtains $\eta = -\frac{\Delta}{2\pi}$, while, at first order,

$$p(\theta) = \frac{-2c_2(1-K)r(\theta) + K\bar{z}\sqrt{1+c_1^2}\sin(\nu-\theta)}{2\pi\Delta\sqrt{1-K}} .$$

By replacing the expression for $r(\theta)$,

$$p(\theta) = C\bar{z} \left[-\frac{2c_2(1-K)}{a_\xi \Delta} \cos(\nu + \xi - \theta) + \sin(\nu - \theta) \right]$$

where

$$C = \frac{K}{2\pi\Delta} \sqrt{\frac{1+c_1^2}{1-K}} .$$

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By finally, recombining the two sinusoidal terms, one finds

$$p(\theta) = a_\gamma C \bar{z} \cos(\nu + \gamma - \theta)$$

where

$$a_\gamma e^{i\gamma} = \frac{-2(1-K)c_2 e^{i\xi}}{a_\xi \Delta} - i .$$

Thus, also $p(\theta)$ is a purely harmonic function and $\tilde{p}(1) = \pi \bar{z} a_\gamma C e^{i(\nu+\gamma)}$. The overall effect of the coupling is finally determined by inserting the expressions for $\tilde{r}(1)$ and $\tilde{p}(1)$ into Eq. (4.12). Since both terms are proportional to $\bar{z}$, we can interpret

$$G := g_z e^{i\delta} := \frac{A}{2} e^{i(\xi+\nu)} + \pi a_\gamma C \sqrt{1-K} e^{i(\gamma+\nu)} ,$$

as the expansion factor of $\bar{z}$ in the presence of a given small modulation of both $\mathcal{C}$ and the probability density. Imposing $G = 1$ finally allows determining a self-consistent solution. More precisely, one can determine the frequency ω of the collective rotation (so far unspecified) and the bifurcation point K . For $c_1 = -2$ and $c_2 = 3$, one obtains $K = K_1 \simeq 0.4123\dots$ and $\omega = \omega_c = -2.5261\dots$, *i.e.*, in agreement with the stability analysis of the splay state. Nevertheless, the amplitude $\bar{z}$ of the order parameter is undetermined at first order. This observation is consistent with the degeneracy exhibited by the Kuramoto-Sakaguchi model at criticality. Moreover, it implies that at least a second harmonic needs to be included in the expansion to obtain a full closure of the equations. This is at variance with systems such as the biharmonic model studied in Chapters 2 and 3, where the shape of the probability density is characterized by the presence of a finite second harmonic from the very beginning. This fact anyway confirms that the presence of second (or higher) harmonics is crucial for the sustainment of SCPS. In the present case such harmonics are spontaneously induced by the modulation of $\mathcal{C}$.

Additionally, the expansion derived in this section implies that, at leading order, the

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coupling function $\mathcal{G}$ can be written as the sum of two terms,

$$\mathcal{G} = 2c_2\sqrt{1-K}r(\theta) - K\bar{z}\sqrt{\frac{1+c_1^2}{1-K}}\sin(\nu-\theta),$$

where r is defined in Eq. (4.13) and $\nu = \arctan c_1$. Since $r(\theta)$ is itself sinusoidal, $\mathcal{G}$ is purely sinusoidal as well. The main difference with the Kuramoto-Sakaguchi model is that here there is an additional indirect dependence of the order parameter through the modulation amplitude $r(\theta)$ of $\mathcal{C}$. At criticality, the difference between the phase of the order parameter and that of $r(\theta)$ is $\xi + \nu = -0.1736$, while the difference with the phase of the probability density is $\gamma + \nu = 0.0523$.

4.4.2 Stability analysis of SCPS

Since $(R_0(\theta), P_0(\theta))$ is a fixed point of Eqs. (4.6) and (4.7), one can easily study the stability of SCPS by determining the eigenvalues of the corresponding linear operator. Let $v(\theta, t)$ and $u(\theta, t)$ denote an infinitesimal perturbation of $R_0(\theta)$ and $P_0(\theta)$, respectively. By inserting $R(\theta, t) = R_0(\theta) + v(\theta, t)$ and $P(\theta, t) = P_0(\theta) + u(\theta, t)$ into Eqs. (4.6, 4.7) and retaining only first order terms, one obtains the linear evolution equations,

$$\begin{aligned} [v(\theta, t)]_t &= v(\theta, t)F^{(v)}(\theta) + [v(\theta, t)]_\theta G^{(v)}(\theta) \\ &\quad + w(t)X^{(v)}(\theta) + \hat{w}(t)Y^{(v)}(\theta) \end{aligned} \quad (4.14)$$

$$\begin{aligned} [u(\theta, t)]_t &= \frac{d}{d\theta} \left[v(\theta, t)F^{(u)}(\theta) + u(\theta, t)G^{(u)}(\theta) \right. \\ &\quad \left. + w(t)X^{(u)}(\theta) + \hat{w}(t)Y^{(u)}(\theta) \right] \end{aligned} \quad (4.15)$$

where

$$w(t) = \int_0^{2\pi} d\theta e^{i\theta} (vP_0 + uR_0) \quad \text{and}$$

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$$\hat{w}(t) = \int_0^{2\pi} d\theta e^{-i\theta} (vP_0 + uR_0) \quad (4.16)$$

are the mean-field contributions in tangent space and

$$\begin{aligned} F^{(v)}(\theta) &:= (1 - K) - 3R_0(\theta)^2 + 2c_2 R'_0(\theta) R_0(\theta) \\ &\quad + \frac{K\bar{z} R'_0(\theta)}{R_0^2(\theta)} \sqrt{1 + c_1^2} \sin(\nu - \theta) , \\ G^{(v)}(\theta) &:= Kc_1 + \omega + c_2 R_0^2 - \frac{K\bar{z}}{R_0(\theta)} \sqrt{1 + c_1^2} \sin(\nu - \theta) , \\ X^{(v)}(\theta) &:= \frac{K\sqrt{1 + c_1^2}}{2} \left(1 - \frac{R'_0(\theta)}{iR_0(\theta)} \right) e^{i(\nu - \theta)} , \\ Y^{(v)}(\theta) &:= \frac{K\sqrt{1 + c_1^2}}{2} \left(1 + \frac{R'_0(\theta)}{iR_0(\theta)} \right) e^{-i(\nu - \theta)} , \\ F^{(u)}(\theta) &:= 2c_2 P_0(\theta) R_0(\theta) + \frac{K P_0(\theta) \bar{z}}{R_0(\theta)^2} \sqrt{1 + c_1^2} \sin(\nu - \theta) , \\ G^{(u)}(\theta) &:= Kc_1 + \omega + c_2 R_0(\theta)^2 - \frac{K\bar{z}}{R_0(\theta)} \sqrt{1 + c_1^2} \sin(\nu - \theta) , \\ X^{(u)}(\theta) &:= -\frac{K P_0(\theta)}{2i R_0(\theta)} \sqrt{1 + c_1^2} e^{i(\nu - \theta)} , \\ Y^{(u)}(\theta) &:= \frac{K P_0(\theta)}{2i R_0(\theta)} \sqrt{1 + c_1^2} e^{-i(\nu - \theta)} . \end{aligned}$$

The equations are better solved in Fourier space. By invoking the Fourier transform, the integrals in the linearized mean-field expression (4.16) can be expanded as a linear combination of $\{\tilde{v}(k, t)\}_{k=-\infty}^{\infty}$ and $\{\tilde{u}(k, t)\}_{k=-\infty}^{\infty}$,

$$w(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{v}(k) \tilde{P}_0(1 - k) + \tilde{u}(k) \tilde{R}_0(1 - k)$$

and

$$\hat{w}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{v}(k) \tilde{P}_0(-1 - k) + \tilde{u}(k) \tilde{R}_0(-1 - k).$$

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Similarly, one can express Eqs. (4.14) and (4.15) in Fourier space. Retaining the terms for each wavelength k one obtains an infinite system of linear equations,

$$[\tilde{v}(k)]_t = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \tilde{v}(j) \left[\tilde{F}^{(v)}(k-j) - ij\tilde{G}^{(v)}(k-j) \right. \\ \left. + \tilde{X}^{(v)}(k)\tilde{P}_0(1-j) + \tilde{Y}^{(v)}(k)\tilde{P}_0(-1-j) \right] \\ + \tilde{u}(j) \left[\tilde{X}^{(v)}(k)\tilde{R}_0(1-j) + \tilde{Y}^{(v)}(k)\tilde{R}_0(-1-j) \right],$$

and

$$[\tilde{u}(k)]_t = \frac{-ik}{2\pi} \sum_{j=-\infty}^{\infty} \tilde{v}(j) \left[\tilde{F}^{(u)}(k-j) + \tilde{X}^{(u)}(k)\tilde{P}_0(1-j) + \tilde{Y}^{(u)}(k)\tilde{P}_0(-1-j) \right] \\ + \tilde{u}(j) \left[\tilde{G}^{(u)}(k-j) + \tilde{X}^{(u)}(k)\tilde{R}_0(1-j) + \tilde{Y}^{(u)}(k)\tilde{R}_0(-1-j) \right].$$

As done in the two previous chapters, it is possible to obtain accurate solutions by truncating the Fourier series at a large enough wavenumber M . The correct eigenvalues are thereby identified as those that are stable against an increase of M . Additionally, the correctness of the selected values has been double checked by integrating the corresponding eigenvectors (see Eqs. (4.14,4.15)). Not unexpectedly, the most relevant eigenvalues (and eigenvectors) do not require large M values.

Figure 4.4(e) shows the resulting spectra for $K = 0.413$ and 0.414 ($M = 400$ modes have been used in the Fourier expansions). Analogously to the splay state, there are basically two sets of eigenvalues, one having strictly negative real parts, while the other corresponding to almost marginally stable directions (though not strictly vanishing as in the splay state). For $K = 0.413$ all the directions are stable, except for the one corresponding to the phase rotation, while for $K = 0.414$ a pair of complex conjugate eigenvalues is present with a positive real part. The corresponding unstable eigenvectors are depicted in Fig. 4.4(c,d).

4.5. Increasing dynamical complexity

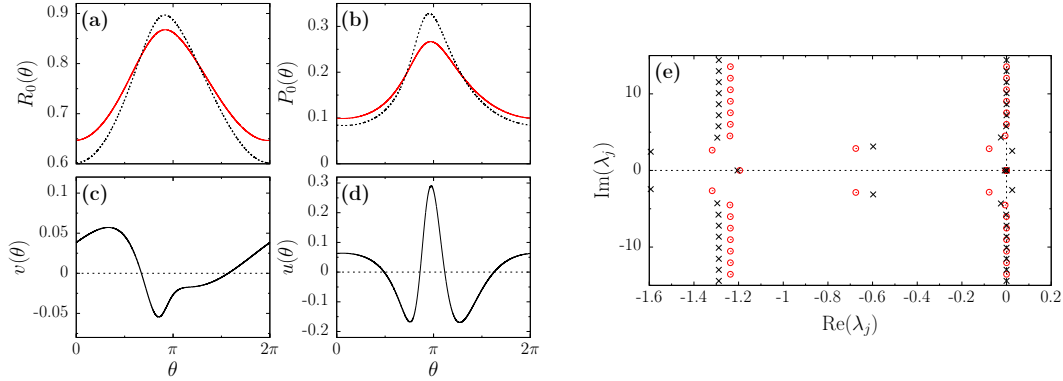


Figure 4.4: (a,b) Shapes of $R_0(\theta)$ and P_0 for $K = 0.413$ (red) and 0.414 (black dashed) obtained by solving Eqs. (4.9,4.10). (c,d) Shape of the unstable eigenfunction $v(\theta)$ and $u(\theta)$ for $K = 0.414$. (e) Eigenvalues resulting from solving the linear equations (4.14) and (4.15) with $M = 400$ Fourier modes for $K = 0.413$ (red circles) and 0.414 (black crosses).

Keeping track of such modes for different K values, one can determine the second bifurcation point $K_2 \simeq 0.413765$. According to this analysis, the bifurcation appears to be a supercritical Hopf bifurcation, which gives rise to a new attractor: the T2 dynamics indicated in Fig. 4.2(a). Nevertheless, as it can be read from the phase diagram, in this new regime the limit cycle does not seem to correspond to a rotation around the unstable fixed point. A detailed analysis of the dynamics of the system close to the bifurcation point shows that the amplitude of the oscillations grows as $\sqrt{K - K_2}$ whereas the frequency does not change significantly. We conjecture that the effect is due to the presence of infinitely many nearly marginal directions, which induce a detachment of the T2 attractor from the plane spanned by the unstable directions of the fixed point.

4.5 Increasing dynamical complexity

An analytical study of time-dependent nonlinear regimes is typically unfeasible. So, from now on, we proceed exclusively on a numerical basis, by relying on the integration of both microscopic and macroscopic equations. The Poincaré section is a qualitative but

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informative tool to understand the dynamical properties of the different regimes observed for larger values of K . Although it was already used in [15], here we consider different observables for reasons that will be clear in a moment. More precisely, we introduce the collective variables

$$Z_n^{(k)} = \left| \int_0^{2\pi} d\psi R(\psi, t_n) P(\psi, t_n) e^{ik\psi} \right|$$

where t_n is the time at which $|\bar{z}|$ reaches a local maximum. This definition is basically an extension of the order parameters typically used to characterize phase oscillators, where the radial contribution R is, for obvious reasons, absent. In phase oscillators, the $Z^{(k)}$ parameters are functionally related to one another whenever the Ott-Antonsen Ansatz is valid [14]. It is therefore instructive to look at their mutual relationship.

In Fig. 4.5(a), we plot $Z_n^{(1)}$ versus K . Since the Poincaré section reduces by one unit the dimensionality of the underlying attractor, a periodic collective dynamics manifests itself as a single point for a given K value. This is indeed what one sees for $K < K_3$, although it should be noticed that the initial section of the curve corresponds to SCPS, where the order parameter is strictly constant. The fuzzy region covered for $K > K_3$ corresponds to a tiny interval where a T3 dynamics initially unfolds, followed by a chaotic regime, analyzed in more quantitative way in the second part of this section.

The panels (b1-b3) of Fig. 4.5 contain various Poincaré section in the T3 region (blue dots) and in the chaotic region (red dots). The points have been obtained by integrating the macroscopic equations. Analogous pictures have been obtained by integrating 32768 oscillators, but significantly blurred by finite-size effects¹. The main message that one learns by comparing the three Poincaré sections is that there is no functional dependence among the first three order parameters, thus suggesting that they are really independent variables – an indirect evidence that the Watanabe-Strogatz theorem does not apply to this context.

How irregular is the regime which settles beyond K_4 ? How does chaos emerge? Ob-

¹The obfuscation of blue points clearly visible in panels (b2) and (b3) of Fig. 4.5 are also artifacts due to the finite accuracy in the determination of the Poincaré section

4.5. Increasing dynamical complexity

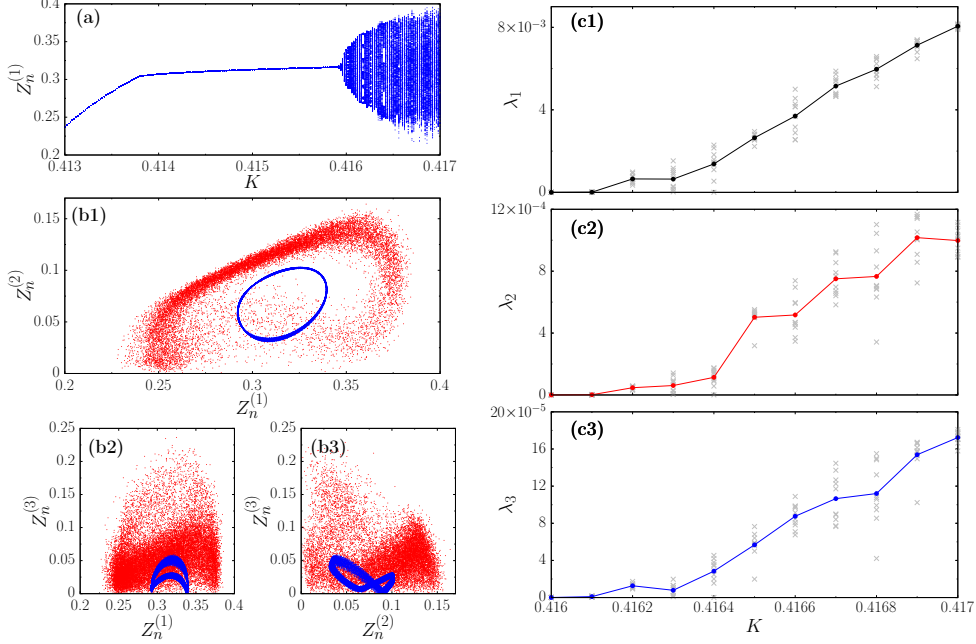


Figure 4.5: (a) Poincaré map of $Z_n^{(1)}$ for different values of the coupling strength K computed using macroscopic simulations. The T1 and T2 regimes appear as stable fixed points, and T3 emerges through a Hopf-bifurcation. Simulations computed along 2×10^4 time units after discarding a transient of 5×10^5 time units. (b1-b3) Poincaré sections resulting from macroscopic simulations for $K = 0.416$ (blue dots) and $K = 0.4165$ (red dots). Simulations computed along 10^5 time units after discarding a transient of 5×10^4 time units. (c1-c3) The three largest Lyapunov exponents of the microscopic system computed over different values of the coupling strength K . Circles indicate the average over ten different realizations, each starting from a different initial condition. Gray crosses show the outcome of the different single realizations. Each simulation consists of $N = 4096$ oscillators, with a quenched disorder of the order of 10^{-14} . The total computation time is of 10^7 time units, of which 10^5 are transient. Simulations computed using fourth order Runge-Kutta with time step $dt = 0.01$. The Gram-Schmidt orthogonalization is invoked every 10 time steps.

taining a clean answer turned out to be arduous. In fact, one has to deal with: (i) long transients; (ii) the spontaneous formation of metastable states; (iii) finite-size effects (in microscopic simulations); (iv) the sporadic formation of highly localized, cluster-like structures (in macroscopic simulations). All of them required much care and long lasting simulations.

We first focus on the computation of the standard, microscopic, Lyapunov exponents. $N = 4096$ turns out to be sufficiently large to ensure negligible finite-size corrections. The

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convergence is nevertheless very slow and a good way to cope with it is by launching simulations from different initial conditions. Moreover, we also add a small heterogeneity of the order of 10^{-14} among the oscillators in order to prevent the formation of spurious clusters due to the finite floating point representation. The parameter dependence of the first three Lyapunov exponents is summarized in Fig. 4.5(c1-c3). The three exponents appear to become positive for approximately the same coupling strength $K_4 \approx 0.4161 \dots$. This is a first indication that we are before a new transition to chaos. It differs from the typical transitions to low-dimensional chaos (period doubling, intermittency, quasiperiodicity), which are accompanied by the change of sign of a single exponent. On the other hand, there is no similarity with the transitions expected in high-dimensional systems, such as spatio-temporal intermittency, where a bunch of exponents becomes positive, whose numerosity is proportional to the system size [20]: a typical signature of extensivity.

A more detailed representation of the overall degree of instability is given in Fig. 4.6, which shows the first 10 Lyapunov exponents deeply inside the chaotic region. All exponents beyond the third one are practically equal to zero. As a result, by virtue of the Kaplan-Yorke formula, the underlying attractor is characterized by a large (possibly infinite in the thermodynamic limit) dimension, in spite of the presence of just three positive exponents: an additional reason to classify this regime as genuinely new.

How does the microscopic instability compare to the macroscopic dynamics? Figure 4.6, also shows the macroscopic Lyapunov exponents, obtained by linearizing the evolution equations (4.2,4.3,4.4) along a generic trajectory. In spite of the large fluctuations (especially those affecting the maximum exponent), the macroscopic spectrum is very similar to the microscopic one. This correspondence is far from obvious and shall be addressed in the final part of this section. Here, we just want to stress that the presence of positive macroscopic exponents shows that we are before a form of collective chaos.

In all mean-field models so far investigated in the literature, collective chaos is accompanied by the instability of the single dynamical units, which can be quantified by

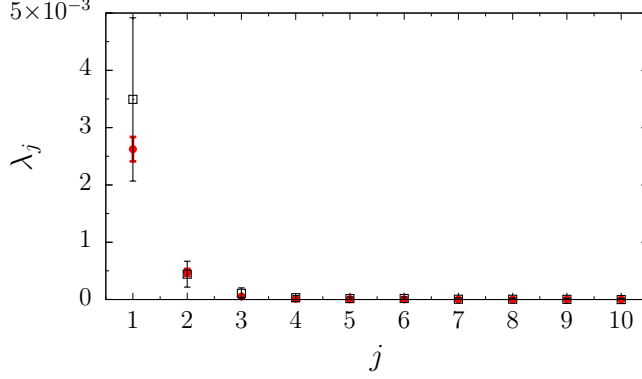


Figure 4.6: The 10 largest Lyapunov exponents for $K = 0.4165$ computed using microscopic (red circles) and macroscopic (open black squares) simulations. The different points show the average over 10 (microscopic) and 50 (macroscopic) simulations starting from different initial conditions, with error bars indicating the standard deviation. Macroscopic simulations computed over 2×10^4 time units after discarding a transient of 5×10^3 . Parameters for microscopic simulations as in figure 4.5(c1-c3).

interpreting the mean field as an external driving force and thereby determining the so-called transverse Lyapunov exponent λ_T . This is the case of logistic maps [21] as well as of Stuart-Landau oscillators in a different parameter region [16]. From now on, we refer to this regime as to *standard collective chaos* (SCC). We now show that the collective dynamics observed beyond T3 has a different nature.

In the present context, λ_T can be determined by linearizing the evolution equation (4.1) under the assumption that $\bar{z}(t)$ is an external forcing,

$$\dot{u} = [1 - K(1 + ic_1)]u - (1 + ic_2)z(2z^*u + zu^*), \quad (4.17)$$

where u denotes an infinitesimal perturbation of z . This equation being two-dimensional (u is a complex variable) is characterized by two (transverse) Lyapunov exponents. The lower exponent is unavoidably negative (in the present case it expresses the stability of deviations from the time-dependent curve $\mathcal{C}$). Less trivial is the value of the largest transverse exponent λ_T , which quantifies the stability of perturbations aligned with the tangent to the curve ($\mathcal{C}$). The best way to describe the underlying phenomenology is by invoking

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ing the standard multifractal formalism, which takes into account the fluctuations of the Lyapunov exponents (see, *e.g.*, the book [22]). Let us start by introducing the generalized Lyapunov exponent

$$\mathcal{L}(q) = \lim_{\tau \rightarrow \infty} \frac{1}{q\tau} \ln \langle |H(\tau)u|^q \rangle$$

where H represents the Jacobian integrated over a time τ (from Eq. (4.17)). $\mathcal{L}(0)$ corresponds to the standard the Lyapunov exponent, while $\mathcal{L}(1)$ corresponds to the topological entropy (in case there is a single positive exponent)². In fact, $\mathcal{L}(1)$ yields the expansion rate of an *arc* of initial conditions initially aligned along the most expanding direction. In the current context, the orientation of the arc corresponds to that of the curve $\mathcal{C}$. Since the curve itself has a fluctuating but finite length, the length of any subsegment does neither grow nor diverge in time, so that $\mathcal{L}(1) = 0$.

Alternatively, the fluctuations of the Lyapunov exponent can be quantified by computing the probability $\mathcal{P}(\Lambda, \tau)$ to observe a finite-time Lyapunov exponent Λ over a time τ and thereby introducing the large deviation function $S(\Lambda)$

$$S(\Lambda) = \lim_{\tau \rightarrow \infty} -\frac{\ln \mathcal{P}(\Lambda, \tau)}{\tau} .$$

The function $S(\Lambda)$ is equivalent to $\mathcal{L}(q)$, the connection between the two representations being given by the Legendre-Fenchel transform [22]

$$q\mathcal{L}(q) = q\Lambda^* - S(\Lambda^*) ,$$

where $q = S'(\Lambda^*)$. In the Gaussian approximation

$$S(\Lambda) = \frac{(\Lambda - \lambda_T)^2}{2D}$$

where λ_T is the standard transverse Lyapunov exponent, while D is the corresponding

²We warn the reader that a different definition is often found in the literature, where $q = 1$ corresponds to the standard Lyapunov exponent. Here we follow the same notations as in [22].

diffusion coefficient defined as

$$D = \lim_{\tau \rightarrow \infty} \tau \left(\overline{\Lambda(\tau)^2} - \lambda_T^2 \right) .$$

As a result, one eventually finds that

$$\mathcal{L}(1) = \lambda_T + \frac{D}{2} . \quad (4.18)$$

Both λ_T and D can be determined from the time evolution of $\gamma(\tau) = \tau\Lambda(\tau)$ ³. In fact, $\gamma(\tau)$ is the logarithm of the expansion factor over a time τ ; it is basically a Brownian motion with a drift velocity λ_T and a diffusion coefficient D . For $K = 0.4165$, upon integrating over 10^7 time units one finds a slightly negative Lyapunov exponent $\lambda_T \approx (-1.7 \pm 0.6) \times 10^{-5}$, while $D \approx (4.4 \pm 0.3) \times 10^{-5}$. As a result, from Eq. (4.18), $\mathcal{L}(1) \approx (0.5 \pm 0.75) \times 10^{-5}$, a value compatible with the expected vanishing exponent. In other words, the fluctuations of the finite-time Lyapunov exponent compensate the slightly negative λ_T and ensure a vanishing expansion factor for the curve length.

Altogether, the message arising from the multifractal analysis is that the (unavoidable) fluctuations of the transverse Lyapunov exponent induce a set of singularities for the corresponding probability density (in the regular SCPS, there are no fluctuations of the Lyapunov exponent which is identically equal to 0). A posteriori, this observation accounts for the difficulties encountered in our simulations of the chaotic phase: in fact, the formation of temporary clusters which so much affect the accuracy of our simulations, irrespective whether they are carried out at the microscopic or macroscopic level, are nothing but a manifestation of the unavoidable presence of singularities, which are intrinsically associated to the self-sustainment of a fluctuating probability density.

We conclude this section by commenting on the similarity between macroscopic and microscopic Lyapunov spectra. The correspondence is unexpected since they arise from

³Here, $\Lambda(\tau)$ denotes the finite-time Lyapunov exponent computed by following a single trajectory from time 0 to time τ .

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two different descriptions of the world. In the microscopic approach, the key variables are the “positions” of the single oscillators, while the macroscopic approach deals with their distribution in phase space. Imagine, for simplicity, to deal with N particles constrained to move along a given curve of fixed length: a virtually infinitesimal microscopic perturbation corresponds to a shift of each particle over a scale that is by definition small compared to the interparticle distance, of the order of $1/N$. On the other hand, a perturbation of the positions for it to be meaningfully interpreted as a perturbation of the corresponding probability density must occur on a scale larger than the statistical fluctuations, which is of order $\sqrt{1/N}$. A priori, there is no guarantee that the linearization of the microscopic equations still hold over such “large scales” (see Ref. [23], for a more detailed discussion of this point). In fact, in typical instances of SCC, macroscopic and microscopic Lyapunov spectra substantially differ from one another. In the toy model of SCC discussed in [23], the collective dynamics is characterized by a single positive macroscopic exponent, while the number of positive exponents is proportional to the number of oscillators, in the microscopic dynamics. The main point of discussion is whether and when some of the microscopic exponents “percolate” to the macroscopic level. In Ref. [16] it is conjectured that this happens whenever the corresponding covariant Lyapunov vector has an extensive nature, but it is still unclear under which conditions this opportunity materializes.

So, what is the difference with the collective chaos discussed in this chapter? The stability analysis of splay states in leaky integrate-and-fire (LIF) neurons can help to shed some light. At the collective level, the state splay corresponds to a trivial stationary homogeneous distribution and its stability can be studied by diagonalizing the corresponding linearized evolution operator. This step was already performed in the early ’90s, determining an analytical expression of the entire spectrum in the weak coupling limit [24]. More recently, the same problem was revisited from the microscopic point of view, analyzing an arbitrary number N of neurons [25], finding that the leading exponents progressively approach the macroscopic ones (upon increasing N), analogously to what observed in Fig. 4.6. The main difference between splay states and SCC is the absence of microscopic chaos and thereby the absence of the statistical fluctuations of size $1/\sqrt{N}$ which would

otherwise represent a sort of “barrier” separating the microscopic from the macroscopic world (see [23] for additional considerations of this point). The maintenance of ordering observed in the collective chaos discussed in this chapter, makes it closer to the splay state than to SCC.

4.6 Conclusions

In this chapter we have developed a formalism that allows characterizing an intermediate regime where amplitude oscillators – while keeping some phase-oscillator properties (*i.e.*, the alignment along a smooth curve) – still exhibit amplitude dynamics. Our analysis deals with Stuart-Landau oscillators, but nothing precludes the application of the same formalism to generic amplitude oscillators, in so far as their evolution remains confined to a smooth curve $\mathcal{C}$.

We confirm that SCPS is a generic regime: a necessary condition for it to be self-sustained is the presence of more than one Fourier harmonics in the effective coupling function, a property that is here induced by the amplitude dynamics. Our formalism allows for an almost analytical characterization of periodic SCPS and, in particular, to determine the bifurcation point, beyond which more complex time-dependent states arise. If SCPS is by itself a non-intuitive regime, since the single oscillators behave without displaying any locking phenomena, the chaotic SCPS described in Section 4.5 is even more so. Each oscillator, under the action of the self-sustained chaotic mean-field, is consistently marginally stable (the transversal generalized Lyapunov exponent defined for $q = 1$ being equal to zero) when the coupling strength is varied. An implication of this observation is that the probability density is unavoidably characterized by the presence of singularities that manifest themselves as temporary clusters.

If and when the transversal Lyapunov exponent becomes positive, a transition to SCC occurs, accompanied by the divergence of the curve $\mathcal{C}$, which would thereby “fill” the phase space (in a fractal way). In the parameter range explored in this chapter, this tran-

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sition is preceded by the onset of a non-conventional incoherent state (*i.e.*, nonuniform distribution characterized by a zero order parameter), which restores a perfectly circular shape of $\mathcal{C}$. It will be worth clarifying the possibly universal mechanisms that may lie behind such a kind of transition.

Finally, the onset of a chaotic SCPS is itself an entirely new phenomenon which involves the simultaneous emergence of more than one positive Lyapunov exponent (actually it looks like three of them). While we could imagine simple mechanisms for the emergence of discontinuous changes (see, for instance, attractor crises), the justification of a continuous transition such as the one discussed in this chapter is by far more intriguing. Last but not least, the question whether chaotic SCPS can be observed in perfect phase oscillators remains open.

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Chapter 5

Dynamical invariance in complex networks

5.1 Introduction

Networks of interacting elements provide a formal representation to model the structure of many complex systems [1, 2], such as social interactions [3], transportation models [4], ecological systems [5], and neuronal functions [6]. The dynamical behavior of such systems cannot be explained as a simple superposition of the dynamics of the single units: one also needs to account for how the different elements interact (type of coupling) and which connectivity structure links the elements (network topology). Dynamical systems on complex networks have been widely studied [7]. However, a general theory to fully resolve the subtle interplay between network structure and ensuing dynamics is still lacking, notwithstanding notable efforts which combine fundamental [8, 9, 10] and applied expertise [11, 12].

The interactions between units in the globally coupled systems considered in the previous chapters was determined by the difference between the value of each unit and the incoming mean-field. The extension of this situation to irregular network topology corresponds to reaction-diffusion systems with identical single node dynamics. In this

case, the coupling between the units always admits a homogeneous state where all the nodes are synchronized. If such equilibrium proves unstable, any arbitrary perturbation grows through the unstable directions giving rise to irregular spatio-temporal patterns [13, 14, 15, 16]. In principle, the specific shape of such patterns depends on both, the dynamics and the network topology, so that it is not easy to predict nor to reproduce. Nevertheless, it is possible to analyze the stability of the homogeneous solution and its dependence on network topology. This irregular behaviour represents then the *functional* response to a given input which can be traced back to the *structural* characteristics of the underlying network [17].

Whereas, in general, a random change on the network topology would lead to a modification of its dynamical response, it is also presumable that a specific outcome is not unique from a particular network, so that similar patterns might arise from different *isodynamic* topologies. The identification of different compatible structures that give rise to the same dynamical behavior represents an important leap forward in the study of complex networks dynamics [18]. For instance, it paves new roads to devise network reconstruction protocols, in cases where more than one network can correspond to the same dynamical output. Upon analyzing the common topological properties that contain the dynamical information of the system, one might overlook the inaccessible details of the topology.

This chapter explores two different methods to, given a specific network, generate a second network isodynamic to the first one, so that they share the same dynamical response. For this purpose, we exploit the idea that just part of the degrees of freedom stored in the network proves significant for the emergence of a specific dynamical behavior [19, 20, 21]. The first method is exact, since it directly acts on the subspace generated by the Laplacian eigenvectors characterizing the stability of the equilibrium state. On the other hand, the second technique relies on a Monte-Carlo algorithm that allows for a neater control in terms of network topology. In brief, the goal of this work is to prove that it is possible to provide two (or more) structurally different complex networks so that inserting the same input on them can lead to the same output.

This work has been developed in collaboration with Giulia Cencetti (University of Florence) and Duccio Fanelli (University of Florence).

5.2 Dispersion relation and pattern formation

Let us consider a generic system composed of N identical entities linked through a complex network. At time t the activity of node j is described by an m -dimensional variable $\mathbf{w}_j(t) \in \mathbb{R}^m$. Starting from a specific initial state $\mathbf{w}_j(0)$, the dynamics of $\mathbf{w}_j$ evolves according to

$$\dot{\mathbf{w}}_j = \mathcal{F}(\mathbf{w}_j) + K \sum_{k=1}^N A_{jk} \mathcal{G}(\mathbf{w}_k - \mathbf{w}_j) \quad j = 1, \dots, N \quad (5.1)$$

where $\mathbf{A}$ denotes the adjacency matrix of the network, and $\mathcal{F} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\mathcal{G} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are generic continuous functions. The first term on the right hand side of the equation describes the self-dynamics of each individual node j , and it is denoted as *reaction term*. The second term instead, defines the interaction of node j with the other nodes of the network: the existence of a coupling is set by the adjacency matrix $\mathbf{A}$, while the interaction shape is established by function $\mathcal{G}$. Our analysis requires two assumptions: (i) there exists an equilibrium $\mathbf{w}_j = \mathbf{w}^*$ for the uncoupled problem $\dot{\mathbf{w}}_j = \mathcal{F}(\mathbf{w}_j)$, and (ii) function $\mathcal{G}$ annihilates in zero ($\mathcal{G}(0) = 0$). The uncoupled equilibrium $\mathbf{w}^*$ can be either a stationary fixed point or a limit cycle. Whatever the case, the second condition ensures that $\mathbf{w}^*$ becomes a *homogeneous solution* of the system (5.1). The case in where $\mathcal{G}$ is a linear function are known as *reaction-diffusion* systems. This constraints can be also generalized in order to include networks of homogeneous Kuramoto-Daido systems with arbitrary coupling functions [22, 23, 24].

At the homogeneous equilibrium $\mathbf{w}^*$ no information is flowing through the network. However, if such state proves unstable, a generic small perturbation $\delta \mathbf{w}$ of $\mathbf{w}^*$ can develop

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to an irregular spatio-temporal pattern. A linearization of equation (5.1) around $\mathbf{w}^*$ provides the time evolution of the perturbation $\delta\mathbf{w} \in \mathbb{R}^{Nm}$ as a system of N linear ordinary differential equations, each of them with dimension m ,

$$\dot{\delta\mathbf{w}} = \mathbf{J}(\mathbf{w}^*)\delta\mathbf{w},$$

where $\mathbf{J} \in \mathbb{R}^{Nm} \times \mathbb{R}^{Nm}$ is the Jacobian matrix of the system. In particular, the m -dimensional equation associated to a perturbation of node j , $j = 1, \dots, N$, reads

$$\dot{\delta\mathbf{w}}_j = \partial_{\mathbf{w}}\mathcal{F}(\mathbf{w}^*)\delta\mathbf{w}_j + K\partial_{\mathbf{w}}\mathcal{G}(\mathbf{0}) \sum_{k=1}^N \left(A_{jk} - k_j^{(\text{in})}\delta_{jk} \right) \delta\mathbf{w}_k = \sum_{k=1}^N \mathbf{J}_{jk}\delta\mathbf{w}_k, \quad (5.2)$$

where $k_j^{(\text{in})}$ stands for the in-degree of node j , and δ_{jk} is the Kronecker delta. Therefore, the stability analysis of $\mathbf{w}^*$ turns into studying the high-dimensional operator $\mathbf{J}(\mathbf{w}^*)$. Nevertheless, it is possible to link the diagonalization of the Jacobian to that of a simpler operator. In the linearized regime, the flow of the quantity $\delta\mathbf{w}_j$ to another node i of the underlying network is described by the term $\Delta_{ij} = A_{ij} - k_i^{(\text{in})}\delta_{ij}$. The resulting $N \times N$ matrix Δ is known as *Laplacian* of the network.

Let us consider the diagonalization of the Laplacian,

$$\sum_k \Delta_{jk}\phi_k^{(\alpha)} = \Lambda^{(\alpha)}\phi_j^{(\alpha)} \quad (5.3)$$

where $\Lambda^{(\alpha)}$ is the α th Laplacian eigenvalue, and $\phi^{(\alpha)} = (\phi_j^{(\alpha)})$ the respective eigenvector. We can now express the perturbation $\delta\mathbf{w}$ on the basis of the Laplacian eigenvectors,

$$\delta\mathbf{w}_j(t) = \sum_{\alpha=1}^N \mathbf{c}_{\alpha}(t)\phi_j^{(\alpha)}, \quad (5.4)$$

where $\mathbf{c}_{\alpha} \in \mathbb{R}^m$. Inserting this expression on (5.2) and making use of (5.3) the time

5.2. Dispersion relation and pattern formation

evolution of δw reads

$$\delta \dot{w}_j = \sum_{\alpha=1}^N \dot{c}_\alpha(t) \phi_j^{(\alpha)} = \partial_w \mathcal{F}(w^*) \sum_{\alpha=1}^N c_\alpha(t) \phi_j^{(\alpha)} + K \partial_w \mathcal{G}(0) \sum_{\alpha=1}^N \Lambda^{(\alpha)} c_\alpha(t) \phi_j^{(\alpha)}.$$

Finally, using the linear independence of the eigenvectors, the previous equation can be decoupled to a m -dimensional system for each α

$$\dot{c}_\alpha(t) = \tilde{J}_\alpha c_\alpha(t) \quad (5.5)$$

where the $m \times m$ matrix $\tilde{J}_\alpha \equiv \partial_w \mathcal{F}(w^*) + K \partial_w \mathcal{G}(0) \Lambda^{(\alpha)}$ is the α -th diagonal block of a new Jacobian matrix $\tilde{J} \in \mathbb{R}^{Nm} \times \mathbb{R}^{Nm}$.

At this point a distinction becomes necessary. If the new Jacobian is constant in time, a simple solution of Eq. (5.5) takes the form $c_\alpha(t) = c_\alpha(0) e^{\lambda^{(\alpha)} t}$ and it is straightforward that in order to admit a non-trivial solution of Eq. (5.5) the determinant of $\tilde{J}_\alpha - \lambda^{(\alpha)} \mathbb{1}_m$ must be zero. From this condition we obtain $\lambda^{(\alpha)} = (\lambda_k^{(\alpha)})$ which configures as the set of eigenvalues of the block $\tilde{J}_\alpha$. If instead the Jacobian is periodic in time due to the limit cycle solution ($\tilde{J}_\alpha(t+T) = \tilde{J}_\alpha(t)$), the Floquet theory [25, 26, 27] comes into play. This latter allows us to express again the temporal dependence of the perturbations δw as an exponential function: $e^{\mu_k^{(\alpha)} t}$, where $\mu_k^{(\alpha)}$ are the *Floquet exponents*, defined by $\mu_k^{(\alpha)} = \log(\rho_k^{(\alpha)})/T$ with $\rho_k^{(\alpha)}$ eigenvalues of the constant matrix B such that $\det(B) = \exp[\int_0^T \text{tr}(\tilde{J}_\alpha(t)) dt]$.

In both, the static and the periodic case, the stability of the system is assessed by the component of respectively $\lambda^{(\alpha)}$ or $\mu^{(\alpha)}$ with maximum real part and its relation with the Laplacian eigenvalues is called the *dispersion relation*. This result is the extension of the dispersion relation defined on a regular (continuous or discrete) medium to the case where the support is irregular. The analogy relies on the identification of the eigenvalues and eigenvectors of the network Laplacian to the Fourier wavelengths and modes used when the support is periodic.

If the dispersion relation of a subportion N_c of the total N eigenvalues $\Lambda^{(\alpha)}$ is positive, then any perturbation would grow through the unstable modes. The resulting irregular spatio-temporal pattern thus represents the unpredictable *response* of the system to a specific *input signal*.

5.3 Topology modification

The dispersion relation provides an effective tool to relate the dynamics of the homogeneous solution to the topology of the network via the network Laplacian. It is however reasonable to suppose that most of the relevant information is stored into the unstable manifold of the homogeneous solution $\mathbf{w}^*$, as we will prove later on. In this section we propose two methods to modify a given discrete support of the system while preserving the relevant directions for the emergence of the pattern.

5.3.1 Eigenmode randomization

The first method for network modification consists of preserving a subportion n of the total N eigenmodes of the original network Laplacian whereas the rest are modified. Let us suppose, without loss of generality, that the subset of modes to be left invariant are the first n . We define the diagonal matrix $\tilde{\Lambda}$, such that

$$\tilde{\Lambda}^{(\alpha)} = \begin{cases} \Lambda^{(\alpha)} & \text{if } \alpha \leq n \\ \Lambda^{(\alpha)} + \delta\Lambda^{(\alpha)} & \text{if } \alpha > n. \end{cases}$$

The corresponding eigenvectors are modified by performing a change of basis

$$\tilde{\Phi} = \Phi \left(\begin{array}{c|c} \mathbb{1}_n & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}_{N-n} \end{array} \right)$$

where $\mathbb{1}_n$ is the identity matrix of dimension n and $\mathbf{R}_{N-n} \in SO(N - n)$ is a random rotation matrix. In order to practically obtain such rotation we perform QR decomposition

5.3. Topology modification

of a random matrix. Taking care of preserving the uniform mode, which corresponds to an identical perturbation acting independently on each node, the transformation

$$\tilde{\Delta} := \tilde{\Phi} \tilde{\Lambda} \tilde{\Phi}^{-1}$$

defines a new Laplacian matrix. Nevertheless, if the original network is directed, the resulting Laplacian might not be real but complex. In Section 5.4.1 we provide more details on how to overcome this situation.

5.3.2 Local rewiring

The previous method does not provide any control whatsoever on the topological modifications introduced in the new Laplacian. For this reason, we propose a secondary route that acts at the level of single nodes. For many network structures, the Laplacian eigenvectors are well localized on the network, *i.e.*, their coordinates in the original vectorial space mostly involve a small subset of nodes, different for each eigenvector [28]. Therefore, modifying some eigenvectors means acting mainly on the connections of a specific subnetwork, and vice versa. With this method we aim to identify and modify links among nodes that are poorly involved on the n -dimensional manifold one wants to preserve.

We rely on a Monte-Carlo algorithm that proceeds as follows. Given the original network, we choose a random non-diagonal entry of the adjacency matrix, A_{ij} . If the entry indicates the existence of a link between node i and node j , such link is removed, otherwise it is created. We then compare the n eigenvalues and eigenvectors of the modified network with those of the original one, and only if they prove similar the change is accepted. To assess the similarity of the original and modified spectra we use the error function

$$E = NE_l + E_q;$$

where

$$E_l = \frac{1}{n} \sum_{\alpha=1}^n |\tilde{\Lambda}^{(\alpha)} - \Lambda^{(\alpha)}|^2$$

and

$$E_q = \frac{1}{n} \sum_{\alpha=1}^n |\langle \tilde{\phi}^{(\alpha)}, \phi^{(\alpha)} \rangle - 1|^2.$$

The factor N is used to ensure that both terms in the expression of E are of the same order. Then, a change on the adjacency matrix is accepted if E is smaller than a certain threshold error, $\tau > 0$. The process is repeated selecting new random entries over the modified adjacency matrix until a desired number of links have been changed or the method fails to detect new entries that lead to small error.

5.4 Results and discussion

5.4.1 Eigenmode randomization

The Ginzburg-Landau model

In order to test the different methods we use the Stuart-Landau oscillator as a reference system for the dynamics of each node. As explained in Chapter 1, this model is the canonical form of the Hopf bifurcation, *i.e.*, it describes a generic limit-cycle at the vicinity of a bifurcation where a fixed point becomes unstable through a pair of complex conjugate eigenvalues crossing the imaginary axis. Therefore, it has been studied in a wide number of applications where periodic oscillatory behavior is relevant.

We first consider an ensemble of Stuart-Landau oscillators occupying the nodes of an undirected network with a diffusive coupling.

$$\dot{w}_j = w_j - (1 + ic_2)|w_j|^2 w_j + (1 + ic_1)K \sum_{k=1}^N \Delta_{jk} w_k$$

where $c_1, c_2, K \in \mathbb{R}$ and $w_j \in \mathbb{C}$. If the interactions were all-to-all, one would recover the system studied in Chapter 4. When considering spatial support, as in this case, this equation takes the name of complex Ginzburg-Landau (CGL) equation [29, 30, 31].

Such system always accepts a homogeneous time-dependent solution, corresponding to the limit-cycle of a single decoupled element of the network $w_j(t) = w^*(t) := e^{-ic_2 t}$. We focus our analysis on this globally synchronous state.

Even though w^* is time-dependent, the Jacobian $\tilde{\mathbf{J}}_\alpha$ is constant in time, and following the procedure explained before, we obtain the dispersion relation characterizing the linear stability analysis of the limit cycle solution,

$$\lambda(-K\Lambda^{(\alpha)}) = K\Lambda^{(\alpha)} - 1 + \sqrt{-c_1^2 K^2 (\Lambda^{(\alpha)})^2 + 2c_1 c_2 K \Lambda^{(\alpha)} + 1}.$$

The Laplacian matrix always has a zero eigenvalue associated to a uniform eigenvector $\phi^{(0)} = (\phi_1^{(0)}, \dots, \phi_N^{(0)}) \propto (1, \dots, 1)$, and all remaining eigenvalues are negative. Thus, $\lambda(0) = 0$ corresponds to the maximum eigenvalue of to the uncoupled equation. Depending on the system parameters, function λ might have a positive part, corresponding to the smaller Laplacian eigenvalues in absolute value. Therefore, a particular network with N nodes might have $N_c < N$ Laplacian eigenvalues associated to unstable modes. From now on, we assume that the Laplacian eigenvalues are sorted in descending order, $\Lambda^{(0)} = 0 > \Lambda^{(1)} > \dots > \Lambda^{(N)}$, so that the N_c unstable modes correspond to the Laplacian eigenvalues with index ranging from 1 to N_c .

Erdős-Rényi networks

We first test the eigenmode randomization method using as original topology an Erdős-Rényi (ER) network with $N = 100$ nodes and average node degree $\langle k \rangle = 3.5$. Figure 5.1(a) shows the dispersion relation of associated to this network for system parameters $K = 1$, $c_1 = 1$, and $c_2 = -3$, (see blue circles). In this case, the system presents $N_c = 44$ unstable directions ($\lambda > 0$), associated to the first (smallest in absolute value) Laplacian eigenvalues, without counting the zero mode associated to the uncoupled system. We integrate the system using as initial condition a randomly generated small perturbation of the homogeneous limit-cycle solution. After a short transient, the system reaches a dynamical regime characterized by an irregular spatio-temporal pattern depicted in Figure 5.1(b)

Chapter 5. Dynamical invariance in complex networks

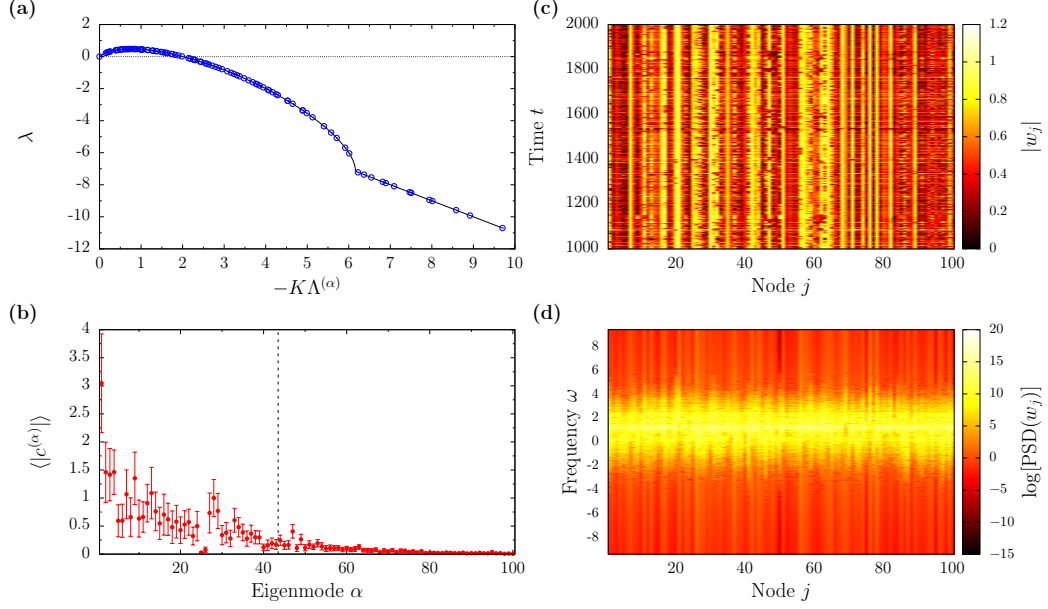


Figure 5.1: Emergence of spatio-temporal patterns for an Erdős-Rényi network with $N = 100$ nodes and $\langle k \rangle = 3.5$. (a): Dispersion relation of the network (blue open circles). (b): Time-averaged modulus of the dynamics expressed in the basis of the Laplacian modes. (c): Time evolution of the modulus of each node coded in color. (d): Logarithm of the power spectral density of each node in color code.

and replicated to Figure 5.2(g) to facilitate future comparisons. Figure 5.1(b) shows the time-averaged modulus of the coefficients $c^{(\alpha)}$ from (5.4), *i.e.*, the coefficients of $w_j(t)$ expressed in the basis of the Laplacian eigenmodes. Thus, it is clear that the modes mostly involved into the patterns are those associated to unstable directions. Throughout this Chapter we focus only on each oscillator modulus, since the frequency of rotation does not seem to contain relevant spatial structure. Indeed, Figure 5.1(d) reports the oscillation frequency of the complex variable w_j , which looks approximately the same for each network node.

We aim to generate a new topology that reproduces such dynamical behavior by leaving invariant a subset of the original network modes. Taking into account that the Laplacian eigenvalues are sorted in descending order, we preserve the first n modes and modify the rest using the eigenmode randomization method. As an illustrative example, Fig-

5.4. Results and discussion

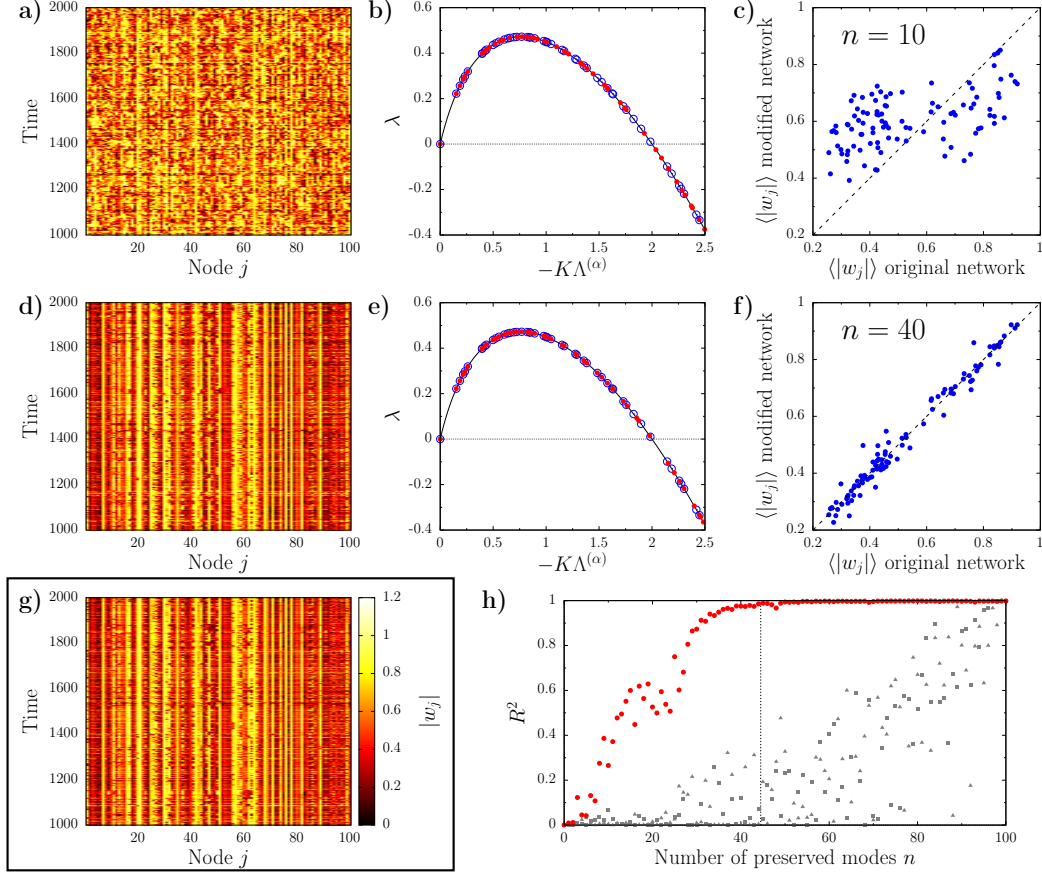


Figure 5.2: Results corresponding to an Erdős-Rényi undirected network with $N = 100$ nodes and average degree $\langle k \rangle = 3.5$. (a-f): Outcome of the implementation of the eigenmode randomization preserving 10 (a-c) and 40 (d-f) modes. (a,d): Time evolution of the modulus for each node of the modified network. (b,e): Section of the dispersion relation λ showing the unstable eigenvalues for the original (red open circles) and modified network (blue closed circles). (c,f): Relation between the time-averaged modulus of each node on the modified network and that of the original network. (g): Time evolution of the modulus of each node of the original network. (h): Squared correlation coefficient between the time-average modulus of each node on the modified network and that of the original network for different number of preserved modes. Purple circles correspond to networks where the first n Laplacian eigenvalues are preserved. Grey squares and grey triangles are the outcome of two different realizations where the n preserved modes of the modified networks are selected at random.

Figure 5.2(a) shows the pattern resulting from a network where only the first $n = 10$ modes of the original topology are preserved and all the others have been randomized. Manifestly,

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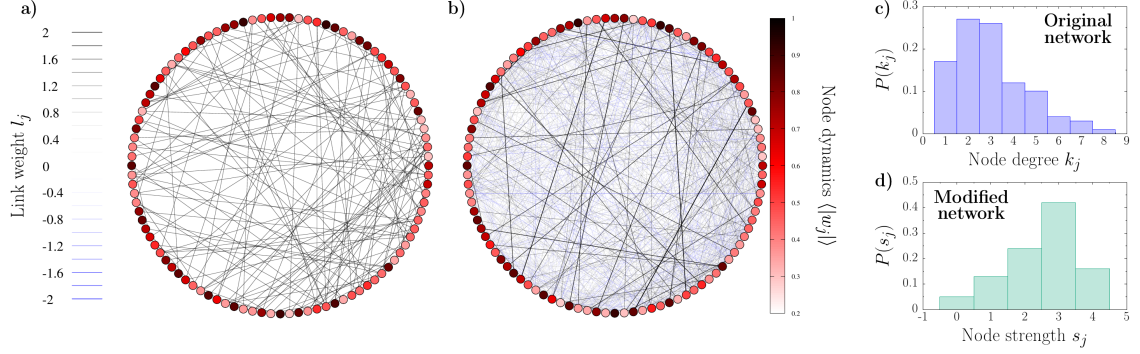


Figure 5.3: (a) Original ER topology. (b) Network resulting from preserving all the unstable modes. The color of the nodes correspond to the time average modulus of W_j , each link thickness is associated to its weight and blue links indicate negative contributions. (c) Node degree distribution of the original network. (d) Node strength of the modified network.

such outcome has little similitude with that of the original network (*cf.* with fig. 5.2(g)). Figure 5.2(b) shows a zoom on the unstable part of the dispersion relation for the modified network (red dots), and that of the original topology (blue open circles). Overall is plausible to explain the disagreement between the dynamical behavior of the two networks from the large difference between the corresponding tangent space of the synchronized solution. Repeating the procedure with $n = 40$ preserved modes instead, leads to the pattern from Fig. 5.2(d). In this case, the behavior of the modified topology resembles much better that of the original network, and observation which is to be traced to the similitude between the dispersion relations of the respective unstable manifolds (see Fig. 5.2(e)).

In order to make the comparison between different patterns more transparent, we compute the time-average modulus of each node, $\langle |w_j| \rangle$ after discarding a transient of 1000 time units. Figures 5.2(c) and (f) show the resulting mean node activity of the modified networks versus that of the original one for $n = 10$ and $n = 40$ respectively. An outcome from two identical patterns would lie exactly on the diagonal, while two completely independent processes would provide a random collection of points. One can then quantify the similitude between the modified and original patterns in terms of squared Pearson correlation coefficient R^2 , which is 0.38 for $n = 10$ and 0.97 for $n = 40$. Figure 5.2(h) shows the outcome of repeating this analysis systematically for increasing number of preserved

modes n . When the modes to be preserved are selected to be the first n , the correlation between the patterns of the modified and original networks increases quickly with n , reaching a very good agreement when approximately all the 44 unstable modes are preserved (see red circles). On the other hand, if the n invariant modes are randomly selected among all the N , even when a large number of directions are maintained, there is no guarantee that the modified network will respond similarly to the original one (see gray triangles and squares).

The eigenmode randomization technique does not provide, *a priori* any information about the structure of the resulting network. Figure 5.3(a) and (b) shows the topologies of the original network and a modified version where all the 44 unstable modes have been preserved. The first obvious difference one can see is the existence of weighted links, including negative ones (see blue edges), which are absent in the original network. The adjacency matrix loses its sparsity and the network becomes highly connected, although most of the new links are very weak. In order to compare the two networks we focus on the *strength* s_j of each node, defined as $s_j = \sum_{m=1}^N A_{jm}$, which extends the concept of node degree to weighted topologies. Figure 5.3(c) and (d) illustrates the degree distribution of the original network, and the strength distribution of the modified topology, respectively. Although the distribution of the new network looks different from the original one, it does not present any particular structure, as expected for a random topology. In fact, the average strength of the new network coincides with the average degree of the initial support.

Scale-Free networks

In order to further investigate the relation between the Laplacian eigenmodes and network topology, we next move to a case where the network has a specific structure. For this purpose we use Scale-Free networks (SF), characterized by a typical power law degree distribution. Figure 5.4(a) show an example of such degree distribution for a network composed of 150 nodes built using the Barabasi-Albert [32] preferential attachment algorithm. Fixing the parameters of the CGL equation as $K = 1$, $c_1 = 1.2$, and $c_2 = -10$, the dispersion relation of this network shows $N_c = 76$ unstable modes. We apply the eigen-

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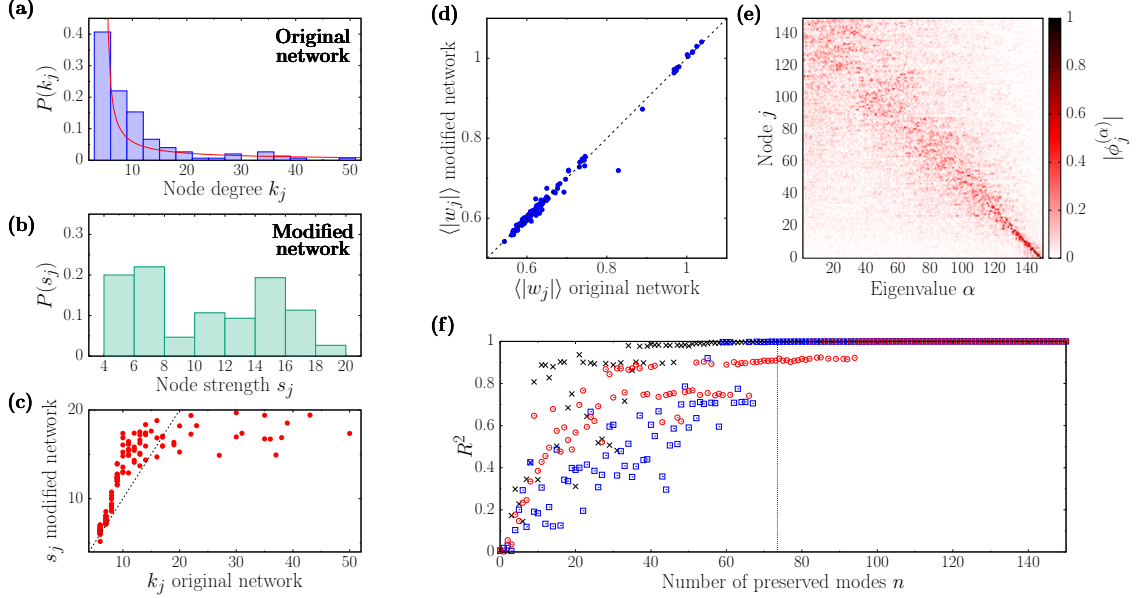


Figure 5.4: Outcome of the eigenmode randomization technique using as original network a scale-free topology with $N = 150$ nodes. (a) Degree distribution of the original network (blue boxes). The continuous red curve corresponds to a numerical fit of the data to an exponential shape $P(k) = ck^{-\beta}$, resulting to $c = 0.23$ and $\beta = 0.85$. (b) Strength distribution on a modified version of the network where the 76 unstable modes have been preserved. (c) Node strength s_j of the modified network with respect to the node degree of the same node k_j in the original network. The black dashed line shows the case $s_j = k_j$ for eye guide. (d) Correlation between the dynamical patterns of the original and modified networks, with resulting squared correlation coefficient $R^2 = 0.99$. (e) Eigenvectors of the original Scale-Free network Laplacian. The eigenvalues are sorted in descending order according to the corresponding Laplacian eigenvalues, and the nodes have been sorted in descending order with respect to their degree. The color represents the absolute value of each eigenvector component. (f) Squared correlation coefficient as the number of preserved modes n increases. Red circles, blue squares and black crosses indicate the results corresponding of three different initial conditions. The vertical dashed line indicates the threshold between unstable and stable modes.

mode randomization technique to generate a new support taking care to preserve all the unstable directions, but none of the stable ($n = N_c$). The patterns resulting from inserting the same perturbation to the synchronized solution on both networks are highly alike, as can be seen from the correlation Figure 5.4(d). Differently from the ER case, here the modulus of each node gets stationary after some transient, so such time-average activity is free from statistical fluctuations.

The topology of the new network is, again, highly connected and involves negative links. Moreover, the node strength distribution (see Fig. 5.4(b)) does not preserve the scale-free structure of the original topology. To identify the nature of this strong changes, in Figure 5.4(c) we plot a one to one comparison between the degree of each node of the original support and the corresponding strength on the new topology. This analysis reveals that nodes with smaller degree on the original setup have a comparable strength after the modification, whereas the hubs become much weaker.

The localization properties of the Laplacian eigenvectors allow us to understand this situation. Inspired by the analysis performed by Hata and Nakao in [28], Figure 5.4(f) depicts the absolute values of the vector components $|\phi_i^{(\alpha)}|$, where nodes are sorted according to their degree in descending order ($k_1 \geq k_2 \geq \dots \geq k_N$), whereas the eigenmode index α follow the usual descending order of the eigenvalues. The almost diagonal behavior indicates that the eigenvectors associated to the eigenvalues with smaller absolute value mostly involve the less connected nodes of the network.

This implies that all the networks generated with the eigenmode randomization procedure will mostly differ from the original one for what concerns the nodes characterized by a large degree k_j . For this reason the characteristic shape of the degree distribution is not preserved in the second network. Nevertheless, in dynamical systems with a different shape of the dispersion relation it might be possible to maintain the tail of the distribution if the stable modes are to be found among the first Laplacian eigenmodes, as will be shown in the next chapter.

Finally, we repeat the procedure of systematically increasing the number of invariant modes n from 1 to 150 and tracking the resulting correlation coefficient, Fig. 5.4(f). For each new network we analyze the dynamical patterns resulting from inserting three different randomly generated perturbations. As for the ER case, the correlation between the dynamics of the original and modified topologies increases quickly with n , being nearly optimal when all the unstable modes are preserved (see vertical dashed line). Neverthe-

less, the choice of the initial condition is clearly relevant: whereas in one case preserving around 40 modes already provides a good agreement (see black crosses), in other situations one might need to preserve also a considerable number of stable directions to reproduce the original pattern (see red circles). This differ from the case of the ER network, where different initial conditions where attracted to the same pattern.

Brusselator model

In order to assess the versatility of the proposed method, in this section we briefly illustrate the case of a different system for the self-dynamics of the single nodes: the Brusselator model. This system displays a stable homogeneous fixed point, with a intrinsically different profile of the associated dispersion relation. [33, 34, 35], and is described by the following equations [36]:

$$\begin{cases} \dot{x}_i = a - (b + d)x_i + cx_i^2y_i + D_x \sum_j \Delta_{ij}x_j \\ \dot{y}_i = bx_i - cx_i^2y_i + D_y \sum_j \Delta_{ij}y_j \end{cases}$$

with $x_i, y_i \in \mathbf{R}$. This system admits as a solution the homogeneous fixed point $(x_i, y_i) = (\frac{a}{d}, \frac{bd}{ac}) \forall i$, whose stability depends on the reaction parameters as well as on the underlying network of connections. By using the same SF reference network than the previous section, and fixing $a = 7$, $b = 10.2$, $c = 1$, $d = 1$, $D_x = 0.1$ and $D_y = 1.4$, the dispersion relation assumes the form depicted in Fig. 5.5(a). The presence of eigenvalues in the positive branch of the curve implies the instability of the uniform fixed point, causing the emergence of irregular patterns.

It is clear that here, differently from the CGL case, the unstable manifold (which contains most of the pattern information) is generated by the modes corresponding to the largest eigenvalues in absolute value. The modes which can be (almost) safely modified in order to generate an isodynamic network consequently correspond to the eigenvalues closest to zero. This amounts to act on the less connected nodes of the network, (see eigenvectors localization in Figure 5.4(f)). The network originated from this procedure

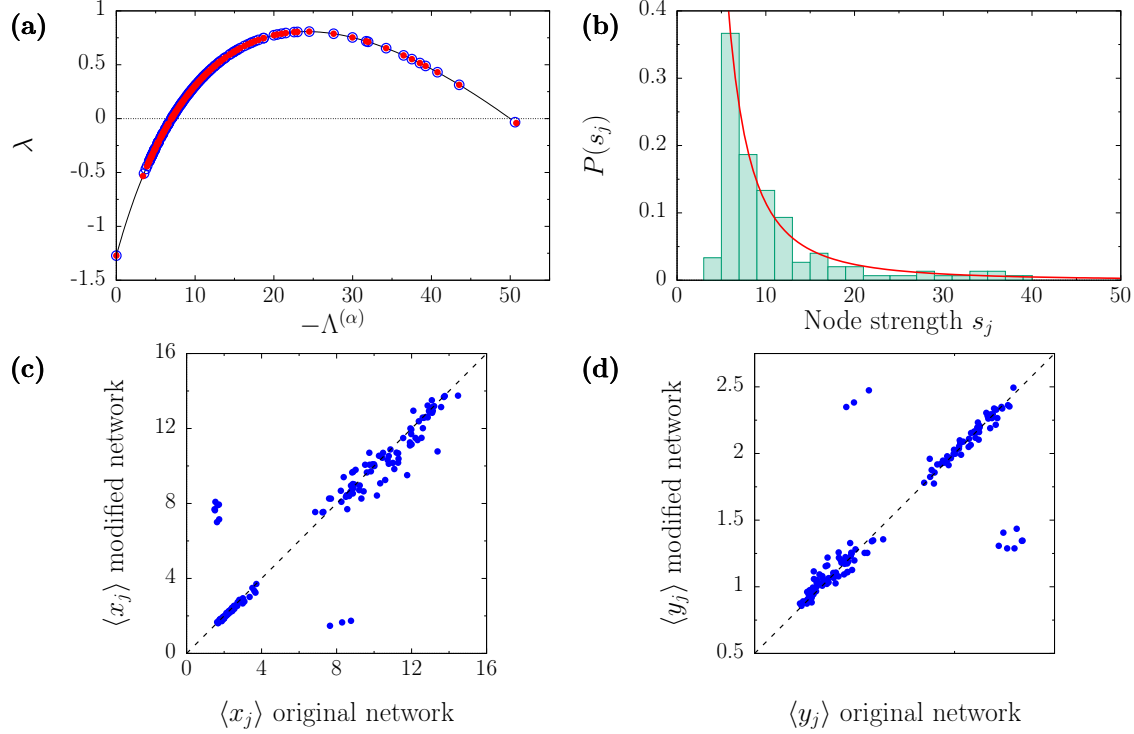


Figure 5.5: Results obtained from the Brusselator model. (a) Dispersion relation of the original network (blue open circles), and that of the modified network (red small circles). (b) Node strength distribution of the modified network (green boxes) and the corresponding fit to a power law $P(k) = ck^{-\beta}$, with $c = 22.3$ and $\beta = -2.29$ (red curve). (c) Time average activity of x_j for the modified and original networks, the corresponding squared coefficient correlation is $R^2 = 0.85$ (d) Time average activity of y_j for the modified and original networks, with $R^2 = 0.74$. The few clearly out-diagonal points correspond to low connected nodes.

preserves then the tail of the characteristic power law shape in the degree distribution, as depicted in Figure 5.5(b). Figures 5.5(c) and (d) prove the similarity between patterns, as obtained by letting the Brusselator model evolve on the original and on the modified version of the SF network, starting from the same initial conditions. Despite some outliers, most the nodes behave in a similar way to those of the original network.

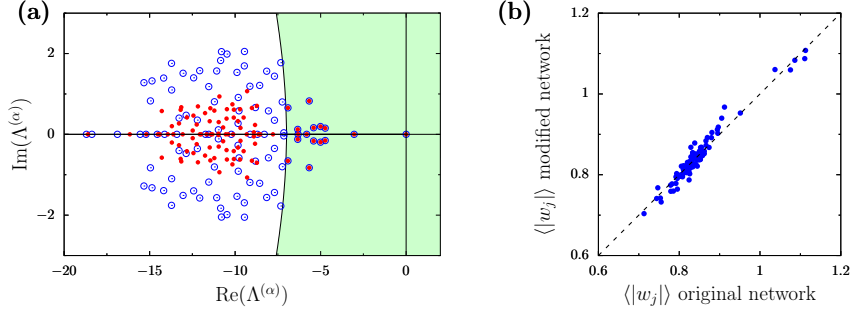


Figure 5.6: (a): Original (red open circles) and modified (blue closed circles) eigenvalues in the complex plane, the coloured area corresponds to the instability region. (b): patterns correlation, corresponding to $R^2 = 0.95$.

Eigenmode randomization for directed networks

The procedure outlined in Section 5.3 requires a special attention when applied to a system where the spatial support is a directed network. The main difference lies in the asymmetry of the Laplacian matrix, so that the eigenvalues and eigenvectors are in general complex. In this case, the unstable eigenmodes can be identified by observing how the eigenvalues are placed in the complex plane. An instability region, corresponding to the positive part of the dispersion relation, can be delineated following the procedure explained before, as it is represented in fig. 5.6. The eigenvalues appearing in this region correspond to the eigenmodes that compose the unstable manifold, which should be preserved to obtain similar patterns in a secondly generated network. All the other eigenvalues and eigenvectors can be modified so as to obtain a new Laplacian matrix. Let us recall that the eigenvalues $\Lambda^{(\alpha)}$, due to the fact that the original Laplacian matrix is real, either are real or are complex and come in conjugate pairs. The same happens for the components of the eigenvectors $\phi^{(\alpha)}$. These characteristics need to be preserved during the randomization in order to obtain a meaningful spectrum for the new Laplacian. However, if the eigenvectors modification is done by using a rotation matrix, the new eigenvectors $\tilde{\phi}^\alpha$ will not in general preserve the complex conjugate couples, which happen to be randomly rotated in the $(N - n)$ -dimensional space. This ultimately implies that the imaginary parts do not

mutually cancel out and that the entries of a Laplacian obtained from

$$\tilde{\Delta} = \tilde{\Phi} \tilde{\Lambda} \tilde{\Phi}^{-1}$$

would be, in general, complex.

Nevertheless, by cutting away the imaginary part of $\tilde{\Delta}$, is possible to obtain yet a third version of the Laplacian, $\hat{\Delta} = \text{Re}(\tilde{\Delta})$, with an associated new set of eigenmodes that still serve to our aim. In fact, the new eigenvectors and eigenvalues are real or complex conjugate and in particular, the n eigenmodes of the original Laplacian to be left invariant are still preserved. In fact, the Laplacian $\tilde{\Delta}$, even if complex, is built so as to correctly maintain the unstable manifold. Hence, its imaginary part (which is eliminated in defining $\hat{\Delta}$) only involves the stable eigenmodes. Therefore, cutting out the imaginary part of $\tilde{\Delta}$ translates into a second randomization of the stable directions, without affecting the unstable manifold where the pattern information is eventually stored.

Let us now proof this statement rigorously. We consider the imaginary part of a generic (i, l) Laplacian entry:

$$\begin{aligned} (\tilde{\Delta}_{\text{Im}})_{il} &= \sum_{\alpha=1}^N \tilde{\Lambda}_{\text{Im}}^{\alpha} [(\tilde{\Phi}_{\text{Re}})_{i\alpha} (\tilde{\Phi}_{\text{Re}}^{-1})_{\alpha l} - (\tilde{\Phi}_{\text{Im}})_{i\alpha} (\tilde{\Phi}_{\text{Im}}^{-1})_{\alpha l}] \\ &\quad + \tilde{\Lambda}_{\text{Re}}^{\alpha} [(\tilde{\Phi}_{\text{Re}})_{i\alpha} (\tilde{\Phi}_{\text{Im}}^{-1})_{\alpha l} + (\tilde{\Phi}_{\text{Im}})_{i\alpha} (\tilde{\Phi}_{\text{Re}}^{-1})_{\alpha l}]. \end{aligned} \quad (5.6)$$

where $\tilde{\Phi}_{\text{Re}}^{-1}$ and $\tilde{\Phi}_{\text{Im}}^{-1}$ are the real and the imaginary part of the eigenvectors matrix. The sum over α can be separated into two sums: one over $\alpha = 1, \dots, n$, the other over $\alpha' = n+1, \dots, N$, the first containing all the unmodified eigenmodes, the second the randomized ones. We shall prove that the first sum is equal to zero. Let us begin by observing that for index α up to n the eigenvalues $\tilde{\Lambda}$ and eigenvectors $\tilde{\phi}$ appearing in the sum can be replaced by the original Λ and ϕ . Considering a conjugate couple labeled with β and γ , one has

Chapter 5. Dynamical invariance in complex networks

$\Lambda^{(\gamma)} = (\Lambda^{(\beta)})^*$, with

$$\begin{aligned}(\phi_{\text{Re}})_{i\gamma} &= (\phi_{\text{Re}})_{i\beta} \\ (\phi_{\text{Im}})_{i\gamma} &= -(\phi_{\text{Im}})_{i\beta}\end{aligned}$$

and

$$\begin{aligned}(\phi_{\text{Re}}^{-1})_{\gamma l} &= (\phi_{\text{Re}}^{-1})_{\beta l} \\ (\phi_{\text{Im}}^{-1})_{\gamma l} &= -(\phi_{\text{Im}}^{-1})_{\beta l}.\end{aligned}$$

This implies the cancellation in pairs of the terms corresponding to complex conjugate couples. For what concerns the remaining terms, corresponding to real eigenvalues and eigenvectors, it is clear that all the terms in (5.6) are automatically zero (remember that $\phi_{jl}^{-1} = \phi_{lj}$). This proves the claim.

In fig. 5.6(a) the eigenvalues of an original random network with $N = 100$ and $\langle k \rangle = 10.2$ are displayed in the complex plane together with those of the modified spectrum. The instability region has been drawn by again using the CGL equation as self-dynamics. Fig. 5.6(b) shows the correlation between time-average modulus patterns obtained from the original and the network generated from the Laplacian $\hat{\Delta}$.

5.4.2 Local rewiring

The second method we present offers the advantage of a larger control on the resulting network topology so that, for instance, one can keep the network binary. On the other hand, one needs to allow for some variability also on the unstable manifold due to the non perfect localization of the Laplacian eigenmodes on a specific subnetwork.

We first apply the local rewiring technique on a ER network with $N = 400$ nodes and average degree $\langle k \rangle = 20$. At each step the tolerance of difference between the unstable manifolds of the original and modified networks is set to $\tau = 10^{-1}$. For system parameters $K = 0.15$, $c_1 = 1$, and $c_2 = -3$, the system displays 64 unstable modes. The algorithm stops after 100 positions on the adjacency matrix have been changed. For comparison purposes, we also created three different networks where 100 entries on the adjacency ma-

trix have been changed totally at random. Blue circles in Fig. 5.7(a) show the correlation between the patterns of the original and the resulting network, whereas the gray symbols correspond to that of the random rewired graphs. The dynamical behavior of the network obtained using the local rewiring method provides a better agreement with the original pattern. We obtain similar results when using SF network generated with the Barabási-Albert algorithm (see Fig. 5.7(b)). Although these results change upon inserting different initial perturbations, the rewiring technique outperform the random modified networks in most of the cases.

In terms of topology changes, Figures 5.7(c) and (d) show the total number of links that have been modified for each node of the network. It is clear that most of the changes correspond to nodes with larger degree which, as shown in the previous section, mostly involve the Laplacian eigenvectors associated to stable directions in the case of the CGL equation.

5.5 Conclusions

In this chapter we have devised two specific strategies of network generation so as to mimic the pattern obtained by a former sample network when both are subject to the same dynamical system. The first method is analytical, and provides isodynamic networks at the cost of changing important topological features of the original graph. The second technique instead makes use of a Monte-Carlo algorithm, allowing for a larger control in terms of network topology, but providing less accurate results. Both methods rely on a preliminary identification of a manifold generated by the Laplacian eigenvectors associated to homogeneous solution instability, which has to be preserved during the modifications.

This work sheds light on the strong dependence between network topology and dynamical patterns. The localization properties of the Laplacian eigenvectors unveil the existence of underlying topological features that lead the dynamical response of the network. In particular, it is possible, in some cases, to identify a subnetwork or a set of nodes that can be

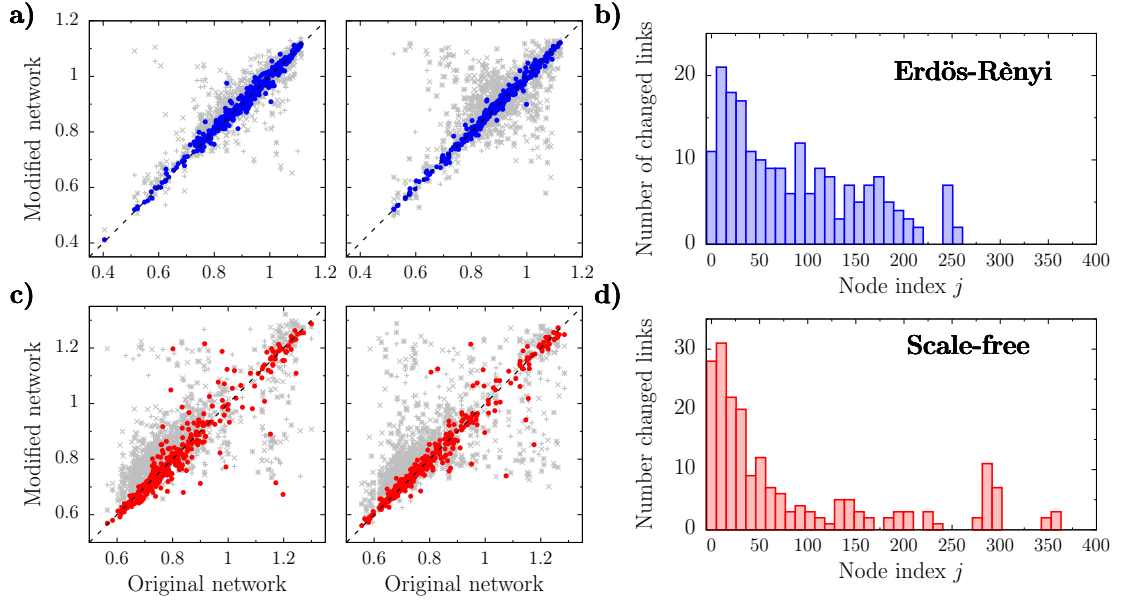


Figure 5.7: Results of the local rewiring method applied to a ER with average degree $\langle k \rangle = 20$ and 64 unstable modes (a,c), and to a SF network with 63 unstable modes (b,d). Both topologies consist of $N = 400$ nodes. At each step the tolerance error of the algorithm is $\tau = 0.1$. In total, the new networks present 100 modified links with respect to the original topologies. (a,b) Correlation between patterns of the original and modified networks. Colored circles correspond to the results obtained using the local rewiring algorithm. Gray pluses, crosses, and stars correspond to three different networks generated by random rewiring as many links as the network outcoming from the local rewiring procedure. The squared coefficient correlations for the modified ER network is 0.98 (blue circles) whereas the randomly rewired networks provide 0.67, 0.62, and 0.94. The modified SF network has $R^2 = 0.86$ (red circles), whereas the random networks provide 0.49, 0.50, and 0.67. (c,d) Number of link changed for each node of the network. The nodes are sorted in descending degree order.

acknowledged as practically irrelevant for pattern formation. It is then clear that distinct networks with differences solely localized on these particular substructures can, under apt conditions, generate the same irregular patterns. As a remarkable example of this situation, we have shown that the SF networks, often taken as reference model to describe many real systems, can be, in the case of the Ginzburg-Landau system, replaced by graphs characterized by highly different structures.

Arguing on this line, the two methods proposed here may be placed as novel schemes

for network generation, with a purely functional angle: instead of building the specific network structure brick by brick, paying attention to local characteristics of nodes, the complex graph is globally established and constrained to yield an *a priori* chosen dynamical output.

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Concluding remarks

Collective synchronization is one of the simplest examples of emerging phenomena in systems composed of a large number of elements. Indeed, the mutual coupling among the oscillators is the main responsible for the phase entrainment of the otherwise freely rotating cells. In this thesis we showed different instances of collective behavior in complex oscillatory systems beyond the paradigm of collective synchronization, shedding thus further light on the way macroscopic states arise from systems defined at the single unit level.

Periodic self-consistent partial synchrony represents a simple case of a collective state that cannot be explained only from the dynamical properties of the single units. This state appears already in a homogeneous phase model with a biharmonic coupling function. Also in this setup, the unexpected regularization of SCPS through the addition of independent random fluctuations at the evolution of each single oscillator further illustrates the robustness of the collective dynamics. Amplitude oscillators provide yet other instances of collective complexity. On one hand, the spontaneous alignment of the oscillators on a smooth closed curve is already another case of spontaneous order arising in a large system. On the other hand, such "global amplitude" adds more harmonics on the "effective coupling" between oscillators, hence leading to different forms of collective behavior, the most intriguing of all being macroscopic high-dimensional chaos arising from non-chaotic units. Finally, the impact of irregular coupling topologies on dynamical function can be tackled with the same philosophy, but distinct methods. The main idea in this case is to characterize the global irregular dynamics in terms of simple observables that involve the

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definition of the network topology. Identifying the modes relevant for the behavior of the system one can indeed modify the architecture of connections while preserving the dynamical outcome.

Although throughout these pages we dealt with simplified mathematical models, the study of large oscillatory systems has an important impact in many areas of research, being neuroscience and important application [1]. Indeed, the study of brain waves and neuronal synchronization mainly involves systems composed of many interconnected oscillatory elements such as neurons or populations of neurons. Realistic neural network models usually contain ingredients that we have not considered here (*e.g.*, time delay, spatial structure or heterogeneity). Nevertheless, we showed that simple setups can already introduce complex forms of collective behavior, providing thus important insight on the impact different characteristics of the systems have on the overall function.

The study of complexity and its mathematical treatment is a growing field which extends beyond oscillatory systems. A well established topic of investigation in this context is the analysis of complex network topologies [2]. Even when no dynamics are considered, in many occasions the question to be answered consist on relating certain local properties of the network with overall structure of the topology. In this framework, although left over from this dissertation for consistency purposes, during this thesis we also worked on a immunization strategy based on the concept of explosive percolation, which produces nearly optimal results in large complex networks with a minimal computation cost [3]. The idea is to identify those nodes whose removal maximizes the fragmentation of the network, hence, it corresponds again to unveil the interplay between local units (the nodes) and global states (fragmentation of the network).

Many questions regarding high-dimensional systems remain open. Through the conclusion sections of each chapter we have reviewed some of them, specific for the topics covered in this thesis. Overall, the emergence of spontaneous collective behavior is a challenging topic of research with impact on many other areas of investigation. It ultimately

addresses some fundamental questions such as how life emerges from simple structures, or how the actions of individuals can affect a society. In fact, the study of complex phenomena is the study of interconnected elements and, therefore, is ubiquitous in real-world. Whereas this thesis does not aim to provide an answer to such general questions, it does represent a step-forward to the general understanding on how the interactions between many elements shape different forms of collective behavior.

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